

ASYMPTOTIC EXPANSIONS FOR A NONLINEAR SINGULARLY
PERTURBED OPTIMAL CONTROL PROBLEM WITH FREE FINAL TIME

By

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An asymptotic expansion method for a nonlinear singularly perturbed two-point boundary value problem arising in a nonlinear optimal control problem with free final time is presented. Assuming that i) the reduced solution satisfies Jacobi's sufficient condition for a local minimum for the reduced problem and ii) the fast subproblem satisfies an eigenvalue criterion, the asymptotic validity for the asymptotic solution is proved via a Banach space fixed point theorem. The proposed asymptotic expansion method is extended to a nonlinear two-point boundary value problem with multiple small parameters.

As a design example, the proposed method is applied to the derivation of a guidance law for an air-to-air bank-to-turn missile. In order for the asymptotic expansion to remain valid during the entire interval, a control weighting adjustment

scheme is used. The resulting guidance law is easily implementable for an on-line computation and gives a good performance under both the deterministic and the stochastic environments.

CHAPTER I

INTRODUCTION

The development of optimal control theory in the early 1960s brought promises of many new areas for applications. In order for a solution of an optimal control problem to be optimal, the solution must satisfy the necessary condition expressed in a two-point boundary value problem (TPBVP). But a TPBVP resulting from application of Pontryagin's minimum principle to an optimal control problem is in general very difficult to solve analytically except some special cases, for example, a linear system with a quadratic cost. Accordingly, many numerical methods have been developed to solve a nonlinear TPBVP effectively [1][2].

However, in a certain class of systems with small parameters multiplying derivatives, the resulting TPBVP is stiff so that a numerical method may fail to solve the TPBVP. This type of system is singularly perturbed and frequently appears in control problem formulations. For example, when one considers i) a dynamic system with small parameters such as small masses and small inertias etc., ii) a dynamic system with a fast actuator and iii) a high gain feedback system, the dynamic system can be formulated as a singularly perturbed system.

One intuitive and widely used approach to design such a control system is to neglect small parameters or fast dynamics. The basic reason for this approach is that one can design a control system from a lower order model. But in this case, the closed loop system designed from the lower order model may be unstable or the performance of the control system may be unacceptable [3]. Another approach

is to seek a trade-off between the lower order system and the high order system involving small parameters or fast dynamics such that the design by a trade-off gives a better performance than that designed from the lower order model and takes less computational burden comparing with the design from the high order model. Many authors have used the singular perturbation method to achieve this goal during the past two decades [4].

The singular perturbation method has been a widely used technique to obtain an approximate solution in the area of an applied mathematics such as fluid mechanics and celestial mechanics. The singular perturbation method has been mainly applied to an optimal control problem since the singular perturbation method was first introduced in a control problem formulation by Kokotovic [5]. But, little has been done for a nonlinear optimal control problem with free final time. In a special class of nonlinear system whose fast state is scalar, some efforts[6] – [9] was made to derive a zeroth order or a first order solution from an optimal control with free final time formulation, but neither an explicit method to construct an asymptotic solution nor its asymptotic validity was treated.

In this dissertation, we have two objectives. One is to develop an explicit method to construct an asymptotic solution in a nonlinear optimal control problem with free final time and fixed end points which plays an important role in a design of an aircraft guidance law. The other is to derive a guidance law for an air-to-air bank-to-turn (BTT) missile.

The major control issues in designing a BTT missile guidance law are i) the BTT missile motion equation is nonlinear, ii) the guidance law must be implementable by on-board computation, iii) the allowable control is limited to a control constraint and

iv) the only target information available are the relative distances in each axis and the closing velocity corrupted by noise. The guidance problem may be formulated as an optimal control problem. Since the missile motion equation is nonlinear, the resulting nonlinear TPBVP is difficult to solve by on-line computation.

The utility of the singular perturbation method when applied to a nonlinear system is that an approximate solution can be obtained by solving several lower dimensional subproblems instead of solving the full problem. Hence, in order to successfully apply the singular perturbation, the problem must be formulated such that the lower dimensional subproblem is solvable without heavy computational burden. In this respect, recently, the singular perturbation method has been successively utilized in obtaining a missile guidance law [6][7]. But, it was observed that the nearly optimal guidance law derived based on the singular perturbation method failed to give a good approximation during the terminal phase in which the time scale separation is not valid [7].

In the application of optimal guidance laws to missile control, feedback guidance laws must be used to compensate for target motion, measurement errors and model errors. However, this means that the optimization problem should be resolved many times as the time-to-go (final time minus initial time) changes. This causes a difficulty with the singular perturbation scheme in that the choice of a performance index which allows time scale separation depends on time-to-go. Therefore, a performance index which will allow good time scale separation and corresponding good approximations for large time-to-go will fail to work for small time-to-go.

In this dissertation, a simple method to specifically overcome the difficulties encountered in the application of the singular perturbation technique to generating

optimal feedback guidance laws over varying time intervals is developed. The actual goal of the guidance law is to minimize the miss distance, but the performance index which is introduced also includes a weighting on the control effort to avoid impulsive controls. The control weighting influences the time scale separation of the solution which is an important assumption in the singular perturbation technique. A method for adjusting the control weighting of the performance index will be presented such that a uniform degree of approximation is obtained as the time-to-go approaches zero. This modification leads to a significant improvement in the final miss distance.

The outline of this dissertation will be as follows.

In Chapter II, the problem is set up and notation necessary for the development for a singularly perturbed free final time optimal control problem is introduced and previous results on singular perturbation theory are reviewed.

In Chapter III, an explicit method to construct an asymptotic solution is described. An asymptotic solution will be constructed in three different time scales.

In Chapter IV, it will be shown that the asymptotic solution constructed in Chapter III is asymptotically valid via a Banach space fixed point theorem.

In Chapter V, a nonlinear system with multiple small parameters will be considered. For a nonlinear system with multiple parameters, we will construct an asymptotic solution by extending the method developed in Chapter III to a multiple parameter case.

In Chapter VI, the main drawback of the currently widely used guidance law will be reviewed and by using the method described in Chapter V, a BTT guidance law up to the first order will be constructed.

In Chapter VII, the simulation results will be illustrated for the case when i) all the measurements are available under non-ideal autopilot/air frame dynamics (deterministic case) and ii) only the bearing only measurements are available under non-ideal autopilot/air frame dynamics (stochastic case).

The final chapter gives conclusions and some suggestions for further works.

CHAPTER II BACKGROUND AND PROBLEM STATEMENT

2.1 Brief Summary of Singular Perturbation Theory

In this section, a mathematical tool developed in order to obtain an approximate solution for a differential equation with a multiple time scale property is reviewed. Singularly perturbed systems and, more generally, multiple time scale systems, often occur due to the presence of small parasitic parameters such as small time constants, masses and capacitors etc. The major advantages of the singular perturbation approach in an analysis of a system are that one can obtain a good approximate solution from a lower order system and avoid an ill conditioning problem resulting from stiffness of the system [5].

First, some preliminary notation will be defined. We are interested in the limit of functions such as $f(\epsilon)$ as ϵ goes to zero. Instead of saying that $f(\epsilon)$ goes to zero at the same rate that ϵ goes to zero, we say $f(\epsilon)$ is order ϵ or write it as $f(\epsilon) = O(\epsilon)$. In general, we put

$$f(\epsilon) = O(g(\epsilon)) \text{ as } \epsilon \rightarrow 0 \text{ if } \lim_{\epsilon \rightarrow 0} \frac{f(\epsilon)}{g(\epsilon)} = k$$

for some finite constant k . Let $\delta_n(\epsilon)$ be a general sequence of functions. $\delta_n(\epsilon)$ is called an asymptotic sequence if it satisfies

$$\lim_{\epsilon \rightarrow 0} \frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} = 0.$$

Given $\sum_{n=0}^{\infty} a_n \delta_n(\epsilon)$, where the a_n does not depend on ϵ and $\delta_n(\epsilon)$ is an asymptotic sequence, we say that this expansion is an asymptotic expansion and write

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\epsilon).$$

We remark that, given an asymptotic sequence $\delta_n(\epsilon)$, the representation of $f(\epsilon)$ in terms of this sequence is unique [10]. In our development, ϵ^n will be used as a $\delta_n(\epsilon)$.

We will define $|x|$, a norm of a vector or a matrix x , as the sum of the absolute values of all the elements of x . Given $f(x(t), y(t), z(t))$, $f_x(t)$ will be used to denote the partial derivative of $f(x(t), y(t), z(t))$ with respect to $x(t)$ unless otherwise specified. Thus,

$$f_x(t) = \frac{\partial f}{\partial x}(x(t), y(t), z(t)).$$

The superscript T denotes the transpose of a vector or a matrix.

For convenience, the following notational convention will be used. If x denotes any vector in $\mathcal{R}^{2n}$ for some integer n , then

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x_1 and x_2 denote the first n and the last n components of x , respectively.

Also, for $2n \times 2n$ matrix A , A_{ij} denotes $n \times n$ submatrix for $1 \leq i, j \leq 2$. Thus,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

First of all, we will consider a simpler problem, namely the singularly perturbed, uncontrolled, autonomous, initial value problem which was treated by numerous

authors [11] – [13].

$$\dot{x} = f(x, z) \quad (2.1)$$

$$\epsilon \dot{z} = g(x, z) \quad (2.2)$$

with the initial condition

$$x(0, \epsilon) = x_0 \quad z(0, \epsilon) = z_0$$

where $x(t, \epsilon)$ and $z(t, \epsilon)$ are n and m dimensional vectors, respectively, and ϵ is a small positive constant which represents the ratio of the time derivative of x and z . The initial conditions x_0 and z_0 are n and m dimensional constant vectors.

We are interested in (2.1) – (2.2) under the assumption that ϵ is small relative to the other parameters of the system. Since $\dot{z}$ is large compared with $\dot{x}$ in this case, we will call x the slow state and z the fast state. It is reasonable to set $\epsilon = 0$ when we want to obtain an approximate solution. Setting ϵ to zero gives

$$\dot{x}_r = f(x_r, z_r) \quad (2.3)$$

$$0 = g(x_r, z_r) \quad (2.4)$$

with the initial condition $x_r(0) = x_0$.

We will call the system (2.1) – (2.2) the full problem and the system (2.3) – (2.4) the reduced problem associated with the full problem. It can be observed that the order of the reduced problem is n . Thus, by perturbing ϵ to zero, the order of the system is changed from $n + m$ to n . For this reason, this type of perturbation is called singular. But in the reduced problem (2.3) – (2.4), in general, z_r does not satisfy the initial condition since z_r is simply a solution of the algebraic equation

(2.4). In order to circumvent this difficulty, the time scale is stretched by introducing a stretched time scale $\tau = \frac{t}{\epsilon}$ and the boundary layer terms $x^L(\tau)$ and $z^L(\tau)$. In the stretched time scale, the full system will be

$$\frac{dx^L}{d\tau} = \epsilon f(x^L, z^L) \quad (2.5)$$

$$\frac{dz^L}{d\tau} = g(x^L, z^L) \quad (2.6)$$

with the initial conditions

$$x^L(0, \epsilon) = x_0 \quad z^L(0, \epsilon) = z_0.$$

The system (2.5) – (2.6) is called the boundary layer problem and the time interval $\tau \in [0, \infty)$ is called the boundary layer. In order to obtain an approximate solution of the problem, it is natural to set $\epsilon = 0$ in (2.5) – (2.6). Then, the resulting system becomes

$$x^L \equiv x_0 \quad (2.7)$$

$$\frac{dz^L}{d\tau} = g(x_0, z^L) \quad (2.8)$$

with the initial condition $z^L(0) = z_0$.

It can be observed that the solution of the reduced problem at $t = 0$ is an equilibrium point of the system (2.7) – (2.8). One crucial condition for the validity of this approximation scheme is the stability of the system (2.7) – (2.8). We will state a theorem taken from Kokotovic's work [14].

Theorem 2.1. Suppose that

1. $f(x, z)$ and $g(x, z)$ are continuously differentiable in some open region Ω of their domain.

2. The reduced problem has a unique solution on some interval $0 \leq t \leq T$.
3. There exists an isolated root $z_r = \phi(x_r)$ of (2.4) in Ω .
4. The solution $z^L(\tau)$ of (2.8) exists on $\tau \in [0, \infty)$ and is unique. Moreover, the system (2.8) is asymptotically stable with respect to the root $\phi(x_r)$ of (2.4).
5. The real parts of the eigenvalues of the Jacobian matrix

$$\frac{\partial g}{\partial z}(x_r, z_r)$$

are negative on $[0, T]$ for $z_r = \phi(x_r)$

Then, for sufficiently small ϵ , the full system (2.1) and (2.2) has a unique solution on $t \in [0, T]$ satisfying the initial conditions $x(0, \epsilon) = x_0$ and $z(0, \epsilon) = z_0$. Furthermore,

$$\lim_{\epsilon \rightarrow 0} x(t, \epsilon) = x_r(t)$$

uniformly on $0 \leq t \leq T$

$$\lim_{\epsilon \rightarrow 0} z(t, \epsilon) = z_r(t)$$

uniformly on any closed subinterval of $0 < t \leq T$.

If a more accurate approximate solution than is obtained in the reduced problem is required, an asymptotic expansion method can be used. This method was pioneered by Vasil'eva [15]. The basic idea is that a solution is sought in the form of a sum of asymptotic expansions which approximates the solution both inside and outside the boundary layer.

For various kinds of TPBVPs, an asymptotic expansion method and its validity was investigated by many authors [16] – [22]. Kokotovic and Sannuti first applied the singular perturbation method to a control system design in order to

obtain a reasonably good approximate solution from a lower order model [5]. After that, Wilde and Kokotovic also applied a singular perturbation method to a linear regulator problem [16]. The necessary conditions for regulator problem for a singularly perturbed linear system leads to a singularly perturbed Riccati equation which also has the multiple time scale property. Therefore, an approximate solution can be obtained by solving a lower order Riccati equation and an algebraic equation. O'Malley also applied the singular perturbation method to a linear regulator problem with a quadratic performance index [17]. He then applied the singular perturbation method to the resulting linear TPBVP.

Hadlock considered a class of nonlinear TPBVP arising in an optimal control problem with a fixed final time and free end points [23]. He showed that the slow states and costates of the full problem converge to the slow states and costates of the reduced problem as ϵ goes to zero, uniformly on the entire closed interval, whereas the fast states and costates of the full problem converge to the fast states and costates of the reduced problem on the open interval $(0, T)$. The lack of convergence of the fast state and costate at the end points results from the fact that the fast states and the costates of the reduced problem which are the solution of an algebraic equation do not satisfy the boundary conditions. But Hadlock did not give an explicit method to construct an asymptotic power series solution.

For a so-called quasi-linear system in which the slow states are nonlinear but the fast states and controls are linear, O'Malley developed a procedure to construct an asymptotic power series solution for an optimal regulator problem [18]. Based on Hadlock's work, Sannuti developed an explicit method to construct an asymptotic power series solution and showed asymptotic validity via a successive approximation

scheme in a class of nonlinear TPBVPs [19]. But Sannuti also restricted the TPBVP to a problem resulting from a fixed final time and free end points quasi-linear optimal control problem. Given a quasi-linear optimal control problem with a fixed final time and a fixed end point problem, Chow developed a method to construct a power series solution [20]. He constructed the boundary layer terms by solving an algebraic Riccati equation.

For a general nonlinear TPBVP resulting from a nonlinear fixed final time free end point optimal control problem, Freedman developed a procedure to construct an asymptotic series solution uniformly valid throughout the interval $[0, T]$ and proved its asymptotic validity [21][22]. Recently, Visser and Shinar developed a procedure to construct an approximate solution in feedback form in a special class of nonlinear optimal control problems with free final time [9]. They assumed that i) a solution from the reduced problem can be obtained in feedback form, ii) the fast state is scalar, iii) there exists no boundary layer at the terminal point and iv) the Hamiltonian is not an explicit function of time. The main motivation for their work is that one can obtain a boundary layer solution by using the fact that the Hamiltonian is identically zero. In this case, one does not have to solve a TPBVP in the boundary layer. But they did not prove the validity of their method.

Breakwell and Shinar developed an alternative method to construct a feedback solution in a class of nonlinear optimal control problem with free final time [24]. They used the Hamilton-Jacobi-Bellman equation instead of using the resulting TPBVP. Their method will be briefly reviewed.

Consider the system

$$\dot{x} = f(x, z, u) \quad x(0) = x_0 \quad (2.9)$$

$$\epsilon \dot{z} = g(x, z, u) \quad z(0) = z_0 \quad (2.10)$$

where x is an n dimensional slow state, z is a scalar fast state and u is an r dimensional control variable.

The objective is to bring the system from the given initial condition to a terminal manifold determined by $x(T)$ while minimizing the performance index

$$J = \int_0^T L(x, z, u) dt$$

where T is an unspecified final time.

Define the optimal return function $V(x(t), z(t), \epsilon)$

$$V(x(t), z(t), \epsilon) = \min_u \int_t^T L(x, z, u) dt'.$$

Then, along an optimal trajectory, the Hamilton-Jacobi-Bellman partial differential equation must be satisfied

$$0 = \min_u (L(x, z, u) + V_x^T f(x, z, u) + \frac{V_z}{\epsilon} g(x, z, u)). \quad (2.11)$$

The optimal control u will be a function of x, z, V_x and $\frac{1}{\epsilon} V_z$.

$$u = U(x, z, V_x, \frac{1}{\epsilon} V_z).$$

By setting $\epsilon = 0$ in (2.9) and (2.10), (2.9) and (2.10) become

$$\dot{x} = f(x, z, u)$$

$$0 = g(x, z, u).$$

Let $z = \phi(x, u)$ be the solution of $0 = g(x, z, u)$. Then the reduced problem is given by

$$\dot{x} = f(x, \phi(x, u), u) \quad (2.12)$$

$$0 = g(x, \phi(x, u), u). \quad (2.13)$$

Assume that

1. The optimal return function

$$V_0(x(t)) = \min_u \int_t^T L(x, \phi(x, u), u) dt$$

can be obtained for the reduced problem.

2. The optimal return function in the full problem has an asymptotic expansion.

Thus,

$$V(x(t), z(t), \epsilon) \sim \sum_{j=0}^{\infty} V_j \epsilon^j$$

Substituting the asymptotic expansion $V(x(t), z(t), \epsilon)$ into (2.11) and matching the coefficients of equal powers of ϵ will give, for $k \geq 1$,

$$V_{k,x} = \frac{\partial V_k}{\partial z} = h_k(x, z)$$

where $h_k(x, z)$ is a function of $x(t), z(t), V_{j,x}$ and $V_{j,z}$ for $0 \leq j \leq k-1$.

Since $h_k(x, z)$ is a known function and $z(t)$ is a scalar, V_k can be obtained by integrating $h_k(x, z)$ with respect to $z(t)$. Hence one can obtain correction terms in the optimal return function recursively as long as integration can be done analytically. Even if this method requires the optimal cost function of the reduced problem and analytical integration, there are many applied problems such that this method can be used.

2.2 Problem Statement

In this section, we will consider a nonlinear singularly perturbed TPBVP arising in an optimal control problem with free final time. The problem formulation is

similar to the problem studied by Breakwell and Shinar [24]; however, we will present a different analytical approach. Consider the following nonlinear system.

$$\dot{x} = \bar{f}(x, z, u) \quad x(0) = x_0 \quad (2.14)$$

$$\epsilon \dot{z} = \bar{F}(x, z, u) \quad z(0) = z_0 \quad (2.15)$$

where x and z are n and m dimensional state vectors, respectively, u is an r dimensional control variable and ϵ is a small positive constant.

The objective is to obtain a control $u(t, \epsilon)$ which minimizes the following scalar performance index with the terminal constraint.

$$J = \phi(T) + \int_0^T L(x, z, u) dt \quad (2.16)$$

$$x(T) = x_T \quad (2.17)$$

where T is free, x_T is prescribed, $L(x, z, u)$ is nonnegative and $\lim_{T \rightarrow \infty} \phi(T) = \infty$.

The functions $\bar{f}$, $\bar{F}$ and L can be functions of ϵ . However, it will be assumed that $\bar{f}$, $\bar{F}$ and L are not functions of ϵ for simplicity. In order to obtain the necessary conditions, we will introduce the Hamiltonian

$$\bar{H}(x, z, p, q, u) = L(x, z, u) + p^T \bar{f}(x, z, u) + \frac{1}{\epsilon} \hat{q}^T \bar{F}(x, z, u) \quad (2.18)$$

where p and $\hat{q}$ are the costate variables corresponding to the state variables x and z .

For convenience of singular perturbation analysis, it is customary to use $q = \frac{1}{\epsilon} \hat{q}$ as the fast costate variables instead of using $\hat{q}$ [17] – [22]. Application of Pontryagin's minimum principle gives the following nonlinear TPBVP which must be satisfied along an optimal trajectory.

$$\dot{p} = -\bar{H}_x(x, z, p, q, u) \quad (2.19)$$

$$\epsilon \dot{q} = -\bar{H}_z(x, z, p, q, u) \quad q(T) = 0 \quad (2.20)$$

$$0 = \bar{H}_u(x, z, p, q, u) \quad (2.21)$$

$$0 = [\phi_t(t) + \bar{H}(x, z, p, q, u)]_{t=T}. \quad (2.22)$$

Now, suppose that $u = U(x, z, p, q)$ satisfies the equation (2.21). Substitution of $u = U(x, z, p, q)$ into (2.14) – (2.15) and (2.19) – (2.22) gives the following $2n+2m$ dimensional TPBVP.

$$\dot{x} = f(x, z, p, q) \quad x(0) = x_0 \quad x(T) = x_T \quad (2.23)$$

$$\dot{p} = g(x, z, p, q) \quad (2.24)$$

$$\epsilon \dot{z} = F(x, z, p, q) \quad z(0) = z_0 \quad (2.25)$$

$$\epsilon \dot{q} = G(x, z, p, q) \quad q(T) = 0 \quad (2.26)$$

$$0 = [\phi_t(t) + H(x, z, p, q)]_{t=T} \quad (2.27)$$

where

$$f(x, z, p, q) = \bar{f}(x, z, p, q, U(x, z, p, q)) \quad (2.28)$$

$$g(x, z, p, q) = -\bar{H}_x(x, z, p, q, U(x, z, p, q)) \quad (2.29)$$

$$F(x, z, p, q) = \bar{F}(x, z, p, q, U(x, z, p, q)) \quad (2.30)$$

$$G(x, z, p, q) = -\bar{H}_z(x, z, p, q, U(x, z, p, q)) \quad (2.31)$$

$$H(x, z, p, q) = \bar{H}(x, z, p, q, U(x, z, p, q)). \quad (2.32)$$

By setting $\epsilon = 0$, the reduced problem can be obtained as follows.

$$\dot{x}_r = f(x_r, z_r, p_r, q_r) \quad (2.33)$$

$$\dot{p}_r = g(x_r, z_r, p_r, q_r) \quad (2.34)$$

$$0 = F(x_r, z_r, p_r, q_r) \quad (2.35)$$

$$0 = G(x_r, z_r, p_r, q_r) \quad (2.36)$$

$$0 = [\phi_i(t) + H(x_r, z_r, p_r, q_r)]_{t=T_r} \quad (2.37)$$

$$x_r(0) = x_0 \quad x_r(T_r) = x_T. \quad (2.38)$$

One can observe that the reduced problem consists of a $2n$ dimensional differential equation and a $2m$ dimensional algebraic equation. In computing the solution of the full problem by a numerical method, the computational burden will be prohibitively large as the order of the full problem increases. Since the full problem is stiff, a small integration step size is required in order for the integration to be numerically stable, further increasing the computational burden. But in the reduced problem, these difficulties may be eliminated and it is possible to obtain an analytic solution for a special class of nonlinear TPBVPs. The most important questions in the study of singular perturbation are

1. Under which conditions will the solution of the full problem approach the solution of the reduced problem as ϵ goes to zero?
2. Under which conditions can we obtain a uniformly valid approximate solution on the interval $[0, T]$ from the reduced problem?
3. How can we make correction terms in order to obtain a more accurate approximate solution?

The above questions for the class of nonlinear TPBVPs described in this section will be investigated in this dissertation.

CHAPTER III ASYMPTOTIC EXPANSIONS

In this chapter, we will construct an asymptotic solution of the full problem (2.23) – (2.27) based on the reduced solution. The construction will be performed in three different time scales. The construction method is similar to the earlier methods [13] [19] [21] for a fixed final time problem except that the final time also can be expanded into a power series in ϵ .

3.1 Basic Approach

Since the solutions of the reduced problem do not satisfy the initial and the terminal conditions, we can expect that the solution of the full problem will have initial and terminal boundary layer terms. Also, we can expect that the optimal final time will be a function of a small parameter ϵ . Moreover, we will assume that all the state, costate variables and the optimal time can be expanded into a power series in ϵ .

It will be assumed that all the variables can be expressed in the following form.

$$x(t, \epsilon) = x^O(t, \epsilon) + \epsilon x^L(\tau, \epsilon) + \epsilon x^R(\sigma, \epsilon) \quad (3.1)$$

$$p(t, \epsilon) = p^O(t, \epsilon) + \epsilon p^L(\tau, \epsilon) + \epsilon p^R(\sigma, \epsilon) \quad (3.2)$$

$$z(t, \epsilon) = z^O(t, \epsilon) + \epsilon z^L(\tau, \epsilon) + \epsilon z^R(\sigma, \epsilon) \quad (3.3)$$

$$q(t, \epsilon) = q^O(t, \epsilon) + \epsilon q^L(\tau, \epsilon) + \epsilon q^R(\sigma, \epsilon) \quad (3.4)$$

where

$$\tau = \frac{t}{\epsilon}, \quad \sigma = \frac{T-t}{\epsilon}.$$

We seek a solution by means of the following asymptotic power series.

$$(x^O(t, \epsilon), p^O(t, \epsilon), z^O(t, \epsilon), q^O(t, \epsilon)) \sim \sum_{j=0}^{\infty} (x_j^O(t), p_j^O(t), z_j^O(t), q_j^O(t)) \epsilon^j \quad (3.5)$$

$$(x^L(\tau, \epsilon), p^L(\tau, \epsilon), z^L(\tau, \epsilon), q^L(\tau, \epsilon)) \sim \sum_{j=0}^{\infty} (x_j^L(\tau), p_j^L(\tau), z_j^L(\tau), q_j^L(\tau)) \epsilon^j \quad (3.6)$$

$$(x^R(\sigma, \epsilon), p^R(\sigma, \epsilon), z^R(\sigma, \epsilon), q^R(\sigma, \epsilon)) \sim \sum_{j=0}^{\infty} (x_j^R(\sigma), p_j^R(\sigma), z_j^R(\sigma), q_j^R(\sigma)) \epsilon^j \quad (3.7)$$

$$T \sim \sum_{j=0}^{\infty} T_j \epsilon^j \quad (3.8)$$

3.2 Outer Expansion

We will seek a solution $(x^O(t, \epsilon), p^O(t, \epsilon), z^O(t, \epsilon), q^O(t, \epsilon))$ which represents the behaviour of the solution of the full problem in the region away from the end points.

Assumptions.

A.1. $f(x, z, p, q), g(x, z, p, q), F(x, z, p, q), G(x, z, p, q), H(x, z, p, q)$ and $\phi(T)$ defined in (2.28) – (2.32) are infinitely differentiable with respect to all their arguments.

A.2. The reduced problem (2.33) – (2.38) has a unique and infinitely differentiable solution on the interval $0 \leq t \leq T_r$.

A.3. The reduced solution satisfies the sufficient conditions [25] for a local minimum. Thus,

1. The convexity condition

2. The normality condition
3. The no conjugate point condition.

A.4. The $2m \times 2m$ matrix $\hat{B}(t)$ is invertible for all $t \in [0, T_r]$, where $\hat{B}(t)$ is defined by

$$\hat{B}(t) = \begin{bmatrix} F_x(t) & F_q(t) \\ G_x(t) & G_q(t) \end{bmatrix}$$

and, for example,

$$F_x(t) = \frac{\partial F}{\partial z}(x_r(t), z_r(t), p_r(t), q_r(t)).$$

Under Assumptions A.1 – A.4, we will construct an asymptotic solution which is valid in a region away from the end points. We will substitute the series (3.5) and (3.8) into the full problem (2.23) – (2.27). The resulting equations may be differentiated j times with respect to ϵ . By setting $\epsilon = 0$, we obtain

For $j = 0$,

$$\dot{x}_0^O = f(x_0^O, z_0^O, p_0^O, q_0^O) \quad (3.9)$$

$$\dot{p}_0^O = g(x_0^O, z_0^O, p_0^O, q_0^O) \quad (3.10)$$

$$0 = F(x_0^O, z_0^O, p_0^O, q_0^O) \quad (3.11)$$

$$0 = G(x_0^O, z_0^O, p_0^O, q_0^O) \quad (3.12)$$

$$0 = \phi_t + H(x_0^O, z_0^O, p_0^O, q_0^O)|_{t=T_0}. \quad (3.13)$$

For $j \geq 1$,

$$\dot{x}_j^O = f_x x_j^O + f_p p_j^O + f_z z_j^O + f_q q_j^O + r_{1j}^O \quad (3.14)$$

$$\dot{p}_j^O = g_x x_j^O + g_p p_j^O + g_z z_j^O + g_q q_j^O + r_{2j}^O \quad (3.15)$$

$$0 = F_x x_j^O + F_p p_j^O + F_z z_j^O + F_q q_j^O + r_{3j}^O \quad (3.16)$$

$$0 = G_x x_j^O + G_p p_j^O + G_z z_j^O + G_q q_j^O + r_{4j}^O \quad (3.17)$$

$$0 = r_{5j}^O T_j + H_x x_j^O + H_p p_j^O + r_{6j}^O|_{t=T_0} \quad (3.18)$$

where r_{ij}^O for $i = 1 \dots 6$ are successively known functions.

Now, we will derive the boundary conditions for the problem (3.9) – (3.13) and (3.14) – (3.18). Assuming that $x^R(\sigma, \epsilon)$ is negligible at $t = 0$, substitution of (3.1) into the initial condition of the slow state of the full problem gives

$$\sum_{j=0}^{\infty} (x_j^O(0) \epsilon^j + x_j^L(0) \epsilon^{j+1}) = x_0.$$

Matching the coefficients of equal powers of ϵ gives

$$x_0^O(0) = x_0 \quad (3.19)$$

$$x_j^O(0) = -x_{j-1}^L(0) \quad j \geq 1. \quad (3.20)$$

Also assuming that $x^L(\tau, \epsilon)$ is negligible at $t = T$, substitution of (3.1) and (3.8) into the terminal condition of the slow state of the full problem gives

$$\sum_{j=0}^{\infty} (x_j^O(T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots) \epsilon^j + x_j^R(0) \epsilon^{j+1}) = x_T.$$

By matching the coefficients of equal powers of ϵ

$$x_0^O(T_0) = x_T \quad (3.21)$$

$$x_j^O(T_0) = -x_{j-1}^R(0) + r_{7j}^O T_j + r_{8j}^O \quad j \geq 1 \quad (3.22)$$

where r_{7j}^O and r_{8j}^O are successively known values.

Definition 3.1. The asymptotic series solution $x_j^O(t)$, $p_j^O(t)$, $z_j^O(t)$, $q_j^O(t)$ and T_j is called the j th order outer solution of the full problem if it satisfies the recursive 2n dimensional TPBVP (3.9) – (3.18) with the boundary conditions (3.19) – (3.22).

We remark that the zeroth order outer solution $(x_0^O, p_0^O, z_0^O, q_0^O, T_0)$ is same as the solution of the reduced problem. Once the zeroth order outer solution is obtained, the higher order solution can be obtained by solving the linear inhomogeneous time varying TPBVP. Since $\hat{B}(t)$ is invertible, $z_j^O(t)$ and $q_j^O(t)$ can be uniquely determined in terms of $x_j^O(t)$ and $p_j^O(t)$ from the algebraic equation (3.16) and (3.17). Assumptions A.2 and A.3 will guarantee the existence and uniqueness of the linear inhomogeneous time varying TPBVP. One powerful method to solve the linear inhomogeneous TPBVP is the backward sweep method [25].

3.3 Boundary Layer Expansion

The outer solution constructed in the previous section does not satisfy all the boundary conditions. In order to obtain a uniformly valid solution throughout the entire interval, boundary layer correction terms known as the left boundary layer solution and the right boundary layer solution will be necessary. Assuming that the right boundary layer terms are negligible in the left boundary layer, from (3.1) – (3.4) the left boundary layer terms must satisfy in time scale τ

$$\frac{dx^L(\tau, \epsilon)}{d\tau} = \left(\frac{dx(t, \epsilon)}{dt} - \frac{dx^O(t, \epsilon)}{dt} \right)_{t=\epsilon\tau} \quad (3.23)$$

$$\frac{dp^L(\tau, \epsilon)}{d\tau} = \left(\frac{dp(t, \epsilon)}{dt} - \frac{dp^O(t, \epsilon)}{dt} \right)_{t=\epsilon\tau} \quad (3.24)$$

$$\frac{dz^L(\tau, \epsilon)}{d\tau} = \epsilon \left(\frac{dz(t, \epsilon)}{dt} - \frac{dz^O(t, \epsilon)}{dt} \right)_{t=\epsilon\tau} \quad (3.25)$$

$$\frac{dq^L(\tau, \epsilon)}{d\tau} = \epsilon \left(\frac{dq(t, \epsilon)}{dt} - \frac{dq^O(t, \epsilon)}{dt} \right)_{t=\epsilon\tau} \quad (3.26)$$

with the initial condition

$$z^L(0, \epsilon) + z^O(0, \epsilon) = z_0. \quad (3.27)$$

We may differentiate (3.23) – (3.27) j times with respect to ϵ and set $\epsilon = 0$.

Then the resulting equations will be given by

For $j = 0$,

$$\begin{aligned} \frac{dx_0^L(\tau)}{d\tau} &= f(x_0^O, z_0^O + z_0^L(\tau), p_0^O, q_0^O + q_0^L(\tau)) \\ &\quad - f(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.28)$$

$$\begin{aligned} \frac{dp_0^L(\tau)}{d\tau} &= g(x_0^O, z_0^O + z_0^L(\tau), p_0^O, q_0^O + q_0^L(\tau)) \\ &\quad - g(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.29)$$

$$\frac{dz_0^L(\tau)}{d\tau} = F(x_0^O, z_0^O + z_0^L(\tau), p_0^O, q_0^O + q_0^L(\tau)) \quad (3.30)$$

$$\frac{dq_0^L(\tau)}{d\tau} = G(x_0^O, z_0^O + z_0^L(\tau), p_0^O, q_0^O + q_0^L(\tau)) \quad (3.31)$$

$$z_0^L(0) = z_0 - z_0^O(0) \quad (3.32)$$

For $j \geq 1$,

$$\frac{dx_j^L(\tau)}{d\tau} = f_z(\tau)z_j^L(\tau) + f_q(\tau)q_j^L(\tau) + r_{1j}^L(\tau) \quad (3.33)$$

$$\frac{dp_j^L(\tau)}{d\tau} = g_z(\tau)z_j^L(\tau) + g_q(\tau)q_j^L(\tau) + r_{2j}^L(\tau) \quad (3.34)$$

$$\frac{dz_j^L(\tau)}{d\tau} = F_z(\tau)z_j^L(\tau) + F_q(\tau)q_j^L(\tau) + r_{3j}^L(\tau) \quad (3.35)$$

$$\frac{dq_j^L(\tau)}{d\tau} = G_z(\tau)z_j^L(\tau) + G_q(\tau)q_j^L(\tau) + r_{4j}^L(\tau) \quad (3.36)$$

$$z_j^L(0) = -z_j^O(0) \quad (3.37)$$

where for example,

$$f_z(\tau) = \frac{\partial f}{\partial z}(x_0^O(0), z_0^O(0) + z_0^L(\tau), p_0^O(0), q_0^O(0) + q_0^L(\tau))$$

$r_{ij}^L(\tau)$ are successively known functions for $i = 1 \dots 4$.

Definition 3.2. The asymptotic series solution $x_j^L(\tau)$, $p_j^L(\tau)$, $z_j^L(\tau)$ and $q_j^L(\tau)$ is called the j th order left boundary layer solution of the full problem if it satisfies the recursive differential equations (3.28) – (3.37). Moreover, we require that there exist some positive constants C and ν such that for all $\tau \in [0, \infty)$,

$$|x_j^L(\tau)| + |p_j^L(\tau)| + |z_j^L(\tau)| + |q_j^L(\tau)| \leq Ce^{-\nu\tau}.$$

We remark that only the initial conditions of the fast state (3.32) and (3.37) are imposed on the differential equations. In order to solve the left boundary layer problem, $q_j^L(0)$ for $j = 0, 1, 2, \dots$ must be so chosen that the resulting solution $z_j^L(\tau)$ and $q_j^L(\tau)$ satisfying (3.30) – (3.31) or (3.35) – (3.36) decay exponentially to zero as τ goes to infinity. Then, $x_j^L(\tau)$ and $p_j^L(\tau)$ can be obtained by simple integration since $\frac{dx_j^L(\tau)}{d\tau}$ and $\frac{dp_j^L(\tau)}{d\tau}$ are known functions of a stretched time scale τ . In order to guarantee the existence of the left boundary layer solution of the full problem, we need the following assumption.

Assumption.

A.5. There exists a $2m \times 2m$ continuously differentiable matrix $M(t)$ for all $0 \leq t \leq T_0$ such that

1.

$$M^{-1}(t)\hat{B}(t)M(t) = \begin{bmatrix} B_1(t) & 0 \\ 0 & -B_1^T(t) \end{bmatrix}$$

2.

$$M_{11}(0) \neq 0 \quad M_{22}(T_0) \neq 0$$

3. $B_1(t)$ has m eigenvalues with negative real parts for each t , $0 \leq t \leq T_0$.

Under Assumptions A.1 and A.5, it was proved by Freedman [22] that for a sufficiently small positive constant δ , the left boundary layer equation (3.28) – (3.37) has a unique solution decaying exponentially to zero as τ goes to infinity for each $j = 0, 1, 2, \dots$ if $|z_j^L(0)| \leq \delta$. Here, a smallness condition of $|z_j^L(0)|$ is inevitable in a general nonlinear context. However, this condition may be eliminated in some cases, for example, quasi-linear systems. The procedure to construct the left boundary layer solution stated in this section is the same as in a fixed final time case. Even if the existence of the left boundary layer solution is guaranteed, it may be prohibitively difficult to solve the left boundary layer equation.

We will proceed to construct the right boundary layer solution. Similarly as in the left boundary layer, in σ time scale the right boundary layer solution must satisfy

$$\frac{dx^R(\sigma, \epsilon)}{d\sigma} = \left(-\frac{dx(t, \epsilon)}{dt} + \frac{dx^O(t, \epsilon)}{dt} \right)_{t=T_0+\epsilon T_1+\epsilon^2 T_2+\dots-\epsilon\sigma} \quad (3.38)$$

$$\frac{dp^R(\sigma, \epsilon)}{d\sigma} = \left(-\frac{dp(t, \epsilon)}{dt} + \frac{dp^O(t, \epsilon)}{dt} \right)_{t=T_0+\epsilon T_1+\epsilon^2 T_2+\dots-\epsilon\sigma} \quad (3.39)$$

$$\frac{dz^R(\sigma, \epsilon)}{d\sigma} = \epsilon \left(-\frac{dz(t, \epsilon)}{dt} + \frac{dz^O(t, \epsilon)}{dt} \right)_{t=T_0+\epsilon T_1+\epsilon^2 T_2+\dots-\epsilon\sigma} \quad (3.40)$$

$$\frac{dq^R(\sigma, \epsilon)}{d\sigma} = \epsilon \left(-\frac{dq(t, \epsilon)}{dt} + \frac{dq^O(t, \epsilon)}{dt} \right)_{t=T_0+\epsilon T_1+\epsilon^2 T_2+\dots-\epsilon\sigma} \quad (3.41)$$

with the initial condition

$$q^R(0, \epsilon) + q^O(T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots, \epsilon) = 0. \quad (3.42)$$

We may differentiate (3.38) – (3.42) j times with respect to ϵ and set $\epsilon = 0$. Then the resulting equations will be given by

For $j = 0$,

$$\begin{aligned} \frac{dx_0^R(\sigma)}{d\sigma} &= -f(x_0^O, z_0^O + z_0^R(\sigma), p_0^O, q_0^O + q_0^R(\sigma)) \\ &\quad + f(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.43)$$

$$\begin{aligned} \frac{dp_0^R(\sigma)}{d\sigma} &= -g(x_0^O, z_0^O + z_0^R(\sigma), p_0^O, q_0^O + q_0^R(\sigma)) \\ &\quad + g(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.44)$$

$$\begin{aligned} \frac{dz_0^R(\sigma)}{d\sigma} &= -F(x_0^O, z_0^O + z_0^R(\sigma), p_0^O, q_0^O + q_0^R(\sigma)) \\ &\quad + F(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.45)$$

$$\begin{aligned} \frac{dq_0^R(\sigma)}{d\sigma} &= -G(x_0^O, z_0^O + z_0^R(\sigma), p_0^O, q_0^O + q_0^R(\sigma)) \\ &\quad + G(x_0^O, z_0^O, p_0^O, q_0^O) \end{aligned} \quad (3.46)$$

$$q_0^R(0) = -q_0^O(0) \quad (3.47)$$

For $j \geq 1$,

$$\frac{dx_j^R(\sigma)}{d\sigma} = -f_z(\sigma)z_j^R(\sigma) - f_q(\sigma)q_j^R(\sigma) + r_{1j}^R(\sigma) \quad (3.48)$$

$$\frac{dp_j^R(\sigma)}{d\sigma} = -g_z(\sigma)z_j^R(\sigma) - g_q(\sigma)q_j^R(\sigma) + r_{2j}^R(\sigma) \quad (3.49)$$

$$\frac{dz_j^R(\sigma)}{d\sigma} = -F_z(\sigma)z_j^R(\sigma) - F_q(\sigma)q_j^R(\sigma) + r_{3j}^R(\sigma) \quad (3.50)$$

$$\frac{dq_j^R(\sigma)}{d\sigma} = -G_z(\sigma)z_j^R(\sigma) - G_q(\sigma)q_j^R(\sigma) + r_{4j}^R(\sigma) \quad (3.51)$$

$$q_j^R(0) = -q_j^O(T_0) + r_{5j}^R \quad (3.52)$$

where r_{ij}^R are successively known functions for $i = 1 \dots 5$ and

$$f_z(\sigma) = \frac{\partial f}{\partial z}(x_0^O(T_0), z_0^O(T_0) + z_0^R(\sigma), p_0^O(T_0), q_0^O(T_0) + q_0^R(\sigma)).$$

Definition 3.3. The asymptotic series solution $x_j^R(\sigma)$, $p_j^R(\sigma)$, $z_j^R(\sigma)$ and $q_j^R(\sigma)$ is called the j th order right boundary layer solution of the full problem if it satisfies the recursive differential equations (3.43) – (3.52). Moreover, we require that there exist some positive constants C and ν such that for all $\sigma \in [0, \infty)$,

$$|x_j^R(\sigma)| + |p_j^R(\sigma)| + |z_j^R(\sigma)| + |q_j^R(\sigma)| \leq Ce^{-\nu\sigma}.$$

As in the left boundary layer, only the initial conditions of the fast costate (3.47) and (3.52) are imposed on the differential equations. In order to solve the right boundary layer problem, $z_j^R(0)$ for $j = 0, 1, 2, \dots$ must be chosen such that the resulting solutions $z_j^R(\sigma)$ and $q_j^R(\sigma)$ decay exponentially to zero as σ goes to infinity. Under Assumptions A.1 and A.5, the right boundary layer problem has a unique right boundary layer solution.

3.4 Discussions

In summary, the procedure to obtain an asymptotic power series solution is given by

1. From the reduced problem (2.33) – (2.38), we obtain a unique reduced solution $x_r(t)$, $p_r(t)$, $z_r(t)$, $q_r(t)$ and T_r .
2. Based on the reduced solution, we obtain $z_0^L(\tau)$, $q_0^L(\tau)$, $z_0^R(\sigma)$ and $q_0^R(\sigma)$ from (3.30) – (3.32) and (3.45) – (3.47).
3. Since the zeroth left and right boundary layer solution are known, $x_0^L(0)$ and $x_0^R(0)$ are computed from (3.28) and (3.43).

$$x_0^L(0) = \int_{\infty}^0 \frac{dx_0^L(\tau)}{d\tau} d\tau$$

$$x_0^R(0) = \int_{\infty}^0 \frac{dx_0^R(\sigma)}{d\sigma} d\sigma$$

4. Since we can determine $x_1^O(0)$ and $x_1^O(T_0)$ from (3.20) and (3.22), the first order outer solution can be obtained from (3.14) – (3.18).
5. Proceeding recursively, the j th order outer solution allows us to solve the j th order left and right boundary layer solution and the boundary condition of the $j + 1$ st order outer problem can be obtained from the j th order left and right boundary layer solution.

After determining the power series solution in each time scale up to the n th order, an approximate solution of the full problem can be made as a sum of the power series solution from (3.1) – (3.4).

CHAPTER IV ASYMPTOTIC VALIDITY

In the previous chapter, we constructed an asymptotic series solution in order to obtain an approximate solution of the full problem. In this chapter, we will show that the construction method described in the previous chapter will give a good approximate solution of the full problem if ϵ and the size of the zeroth order boundary layer jumps in the fast variables are sufficiently small.

It is convenient to make the following abbreviations:

$$y(t) = \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \quad w(t) = \begin{bmatrix} z(t) \\ q(t) \end{bmatrix}.$$

Thus, $y(t)$ represents the slow state and the slow costate and $w(t)$ represents the fast state and the fast costate.

For any integer $N \geq 0$, we define the partial sums by

$$\begin{bmatrix} \bar{y}_N^O \\ \bar{w}_N^O \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^N y_j^O(t) \epsilon^j \\ \sum_{j=0}^N w_j^O(t) \epsilon^j \end{bmatrix} \quad (4.1)$$

$$\begin{bmatrix} \bar{y}_N^L \\ \bar{w}_N^L \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{N-1} y_j^L(\tau) \epsilon^j \\ \sum_{j=0}^N w_j^L(\tau) \epsilon^j \end{bmatrix} \quad (4.2)$$

$$\begin{bmatrix} \bar{y}_N^R \\ \bar{w}_N^R \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{N-1} y_j^R(\sigma) \epsilon^j \\ \sum_{j=0}^N w_j^R(\sigma) \epsilon^j \end{bmatrix} \quad (4.3)$$

$$\begin{bmatrix} \bar{y}_N \\ \bar{w}_N \end{bmatrix} = \begin{bmatrix} \bar{y}_N^O(t) + \epsilon \bar{y}_N^L(\tau) + \epsilon \bar{y}_N^R(\sigma) \\ \bar{w}_N^O(t) + \epsilon \bar{w}_N^L(\tau) + \epsilon \bar{w}_N^R(\sigma) \end{bmatrix} \quad (4.4)$$

$$\bar{T}_N = \sum_{j=0}^N T_j \epsilon^j. \quad (4.5)$$

Let $y(t, \epsilon)$, $w(t, \epsilon)$ and $T(\epsilon)$ be the solution of the full problem. Then $y(t, \epsilon)$, $w(t, \epsilon)$ and $T(\epsilon)$ can be expressed by

$$\begin{bmatrix} y(t, \epsilon) \\ w(t, \epsilon) \end{bmatrix} = \begin{bmatrix} \bar{y}_N(t) \\ \bar{w}_N(t) \end{bmatrix} + \epsilon^N \begin{bmatrix} y^*(t) \\ w^*(t) \end{bmatrix} \quad (4.6)$$

$$T(\epsilon) = \bar{T}_N + \epsilon^N \gamma^*$$

$$\sigma = \frac{T(\epsilon) - t}{\epsilon} = \frac{\bar{T}_N + \epsilon^N \gamma^* - t}{\epsilon}$$

where $\epsilon^N y^*$, $\epsilon^N w^*$ and $\epsilon^N \gamma^*$ denote the remainder terms.

The objective in this chapter is to prove that i) the solution of the full problem $y(t, \epsilon)$, $w(t, \epsilon)$ and $T(\epsilon)$ exists under Assumptions A.1 – A.5 described in Chapter II and ii) $y(t, \epsilon)$, $w(t, \epsilon)$ and $T(\epsilon)$ approach the asymptotic solution $\bar{y}_N$, $\bar{w}_N$ and $\bar{T}_N$ as ϵ^{N+1} goes to zero. The principal tool to be used in the proof is a Banach space fixed point theorem [26].

4.1 Nonlinear TPBVP for the Remainder Terms

We will derive a nonlinear TPBVP governing the remainder terms y^* , w^* and γ^* . One can see that for sufficiently small ϵ there exists $\hat{T} > \bar{T}_N$ such that Assumption A.5 also holds for an interval $[0, \hat{T}]$. Thus, it will be assumed that $B_1(t)$ defined in Assumption A.5 has m eigenvalues with real parts $\leq -2\nu < 0$ for each $t \in [0, \hat{T}]$. We also assume that the outer solution is extendable such that the outer solution is defined throughout the compact interval $[0, \hat{T}]$.

Using (4.4) and differentiation of (4.6) shows that the remainder terms must satisfy

$$\dot{\bar{y}}_N^O + \frac{d\bar{y}_N^L}{d\tau} - \frac{d\bar{y}_N^R}{d\sigma} + \epsilon^N \dot{y}^* = u(y, w) \quad (4.7)$$

$$\epsilon \dot{\bar{w}}_N^O + \frac{d\bar{w}_N^L}{d\tau} - \frac{d\bar{w}_N^R}{d\sigma} + \epsilon^{N+1} \dot{w}^* = U(y, w) \quad (4.8)$$

where

$$u(y, w) = \frac{dy(t, \epsilon)}{dt} = \begin{bmatrix} f(x, z, p, q) \\ g(x, z, p, q) \end{bmatrix}$$

$$U(y, w) = \epsilon \frac{dw(t, \epsilon)}{dt} = \begin{bmatrix} F(x, z, p, q) \\ G(x, z, p, q) \end{bmatrix}.$$

By the construction of the outer solution in the outer problem (3.9) – (3.18), $\bar{y}_N^O$ and $\bar{w}_N^O$ must satisfy

$$\begin{aligned} \dot{\bar{y}}_N^O &= \left(u(\bar{y}_\infty^O, \bar{w}_\infty^O) \right)_N \\ &= u(\bar{y}_N^O, \bar{w}_N^O) + O(\epsilon^{N+1}) \end{aligned} \quad (4.9)$$

where $(\cdot)_N$ represents all the terms containing ϵ^k for $0 \leq k \leq N$.

Similarly,

$$\epsilon \dot{\bar{w}}_N^O = U(\bar{y}_N^O, \bar{w}_N^O) + O(\epsilon^{N+1}). \quad (4.10)$$

We introduce the size of the j th order boundary layer jumps δ_j such that

$$|w_j^L(0)|, |w_j^R(0)| \leq \delta_j \quad 0 \leq j \leq N.$$

Also, we define δ by

$$\delta = \max_{0 \leq j \leq N} \delta_j.$$

From the construction of the left boundary layer solution in the left boundary layer problem (3.28) – (3.37),

$$\begin{aligned}\frac{d\bar{y}_N^L}{d\tau} &= \left(u(\bar{y}_\infty^O + \epsilon\bar{y}_\infty^L, \bar{w}_\infty^O + \bar{w}_\infty^L) - u(\bar{y}_\infty^O, \bar{w}_\infty^O) \right)_{N-1} \\ &= u(\bar{y}_N^O + \epsilon\bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) - u(\bar{y}_N^O, \bar{w}_N^O) + O(\epsilon^N \delta e^{-\nu\tau})\end{aligned}\quad (4.11)$$

$$\begin{aligned}\frac{d\bar{w}_N^L}{d\tau} &= \left(U(\bar{y}_\infty^O + \epsilon\bar{y}_\infty^L, \bar{w}_\infty^O + \bar{w}_\infty^L) - U(\bar{y}_\infty^O, \bar{w}_\infty^O) \right)_N \\ &= U(\bar{y}_N^O + \epsilon\bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) - U(\bar{y}_N^O, \bar{w}_N^O) + O(\epsilon^{N+1} \delta e^{-\nu\tau}).\end{aligned}\quad (4.12)$$

In the right boundary layer, the construction (3.38) – (3.41) gives

$$\begin{aligned}\frac{d\bar{y}_N^R}{d\sigma} &= \left(-u(\bar{y}_N^O(\bar{T}_N - \epsilon\sigma) + \epsilon\bar{y}_N^R, \bar{w}_N^O(\bar{T}_N - \epsilon\sigma) + \bar{w}_N^R) \right. \\ &\quad \left. + u(\bar{y}_N^O(\bar{T}_N - \epsilon\sigma), \bar{w}_N^O(\bar{T}_N - \epsilon\sigma)) \right)_{N-1} \\ &= \left(-u(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon\bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \right. \\ &\quad \left. + u(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) \right)_{N-1} \\ &= -u(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon\bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \\ &\quad + u(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) + O(\epsilon^N \delta e^{-\nu\sigma})\end{aligned}\quad (4.13)$$

$$\begin{aligned}\frac{d\bar{w}_N^R}{d\sigma} &= -U(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon\bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \\ &\quad + U(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) + O(\epsilon^{N+1} \delta e^{-\nu\sigma}).\end{aligned}\quad (4.14)$$

Substitution of (4.9) – (4.14) into (4.7) and (4.8) gives

$$\begin{aligned}\epsilon^N \dot{y}^* &= u(\bar{y}_N^O + \epsilon\bar{y}_N^L + \epsilon\bar{y}_N^R + \epsilon^N y^*, \bar{w}_N^O + \bar{w}_N^L + \bar{w}_N^R + \epsilon^N w^*) \\ &\quad - u(\bar{y}_N^O + \epsilon\bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) + u(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) \\ &\quad - u(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon\bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R)\end{aligned}$$

$$+O(\epsilon^{N+1}) + O(\epsilon^N \delta e^{-\nu\tau}) + O(\epsilon^N \delta e^{-\nu\sigma}) \quad (4.15)$$

$$\begin{aligned} \epsilon^{N+1} \dot{w}^* &= U(\bar{y}_N^O + \epsilon \bar{y}_N^L + \epsilon \bar{y}_N^R + \epsilon^N y^*, \bar{w}_N^O + \bar{w}_N^L + \bar{w}_N^R + \epsilon^N w^*) \\ &\quad - U(\bar{y}_N^O + \epsilon \bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) + U(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) \\ &\quad - U(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon \bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \\ &\quad + O(\epsilon^{N+1}) + O(\epsilon^{N+1} \delta e^{-\nu\tau}) + O(\epsilon^{N+1} \delta e^{-\nu\sigma}). \end{aligned} \quad (4.16)$$

We rewrite the above equations by splitting the right hand sides of (4.15) and (4.16) into linear parts and the remaining nonlinear parts.

$$\dot{y}^* = u_y y^* + u_w w^* + u^*(y^*, w^*, \gamma^*, \epsilon) \quad (4.17)$$

$$\epsilon \dot{w}^* = U_y y^* + U_w w^* + U^*(y^*, w^*, \gamma^*, \epsilon) \quad (4.18)$$

where

$$\begin{aligned} \epsilon^N u^*(y^*, w^*, \gamma^*, \epsilon) &= u(\bar{y}_N^O + \epsilon \bar{y}_N^L + \epsilon \bar{y}_N^R + \epsilon^N y^*, \bar{w}_N^O + \bar{w}_N^L + \bar{w}_N^R + \epsilon^N w^*) \\ &\quad - u(\bar{y}_N^O + \epsilon \bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) + u(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) \\ &\quad - u(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon \bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \\ &\quad - \epsilon^N u_y y^* - \epsilon^N u_w w^* \\ &\quad + O(\epsilon^{N+1}) + O(\epsilon^N \delta e^{-\nu\tau}) + O(\epsilon^N \delta e^{-\nu\sigma}) \end{aligned} \quad (4.19)$$

$$\begin{aligned} \epsilon^N U^*(y^*, w^*, \gamma^*, \epsilon) &= U(\bar{y}_N^O + \epsilon \bar{y}_N^L + \epsilon \bar{y}_N^R + \epsilon^N y^*, \bar{w}_N^O + \bar{w}_N^L + \bar{w}_N^R + \epsilon^N w^*) \\ &\quad - U(\bar{y}_N^O + \epsilon \bar{y}_N^L, \bar{w}_N^O + \bar{w}_N^L) + U(\bar{y}_N^O(t - \epsilon^N \gamma^*), \bar{w}_N^O(t - \epsilon^N \gamma^*)) \\ &\quad - U(\bar{y}_N^O(t - \epsilon^N \gamma^*) + \epsilon \bar{y}_N^R, \bar{w}_N^O(t - \epsilon^N \gamma^*) + \bar{w}_N^R) \\ &\quad - \epsilon^N U_y y^* - \epsilon^N U_w w^* \end{aligned}$$

$$+O(\epsilon^{N+1}) + O(\epsilon^{N+1}\delta e^{-\nu\tau}) + O(\epsilon^{N+1}\delta e^{-\nu\sigma}) \quad (4.20)$$

where u_y is defined by

$$u_y = \frac{\partial u}{\partial y}(\bar{y}_N^O, \bar{w}_N^O, \epsilon)$$

and u_w, U_y and U_w are also similarly defined.

The boundary conditions for the remainder terms can be determined as follows.

The initial x is given by

$$\begin{aligned} x_0 &= \bar{x}_N(0) + \epsilon^N x^*(0) \\ &= \bar{x}_N^O(0) + \epsilon \bar{x}_N^L(0) + \epsilon \bar{x}_N^R\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) + \epsilon^N x^*(0). \end{aligned}$$

Since $\bar{x}_N^O(0) + \epsilon \bar{x}_N^L(0) = x_0$ by construction,

$$x^*(0) = -\epsilon^{-N+1} \bar{x}_N^R\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right). \quad (4.21)$$

The boundary condition at the terminal time is obtained by considering the final x .

$$\begin{aligned} x_T &= \bar{x}_N^O(\bar{T}_N + \epsilon^N \gamma^*) + \epsilon \bar{x}_N^L\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) + \epsilon \bar{x}_N^R(0) + \epsilon^N x^*(\bar{T}_N + \epsilon^N \gamma^*) \\ &= \bar{x}_N^O(\bar{T}_N) + \epsilon \bar{x}_N^R(0) + \epsilon \bar{x}_N^L\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) \\ &\quad + \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\bar{x}}_N^O(s) ds + \epsilon^N x^*(\bar{T}_N + \epsilon^N \gamma^*). \end{aligned}$$

Since $\bar{x}_N^O(\bar{T}_N) + \epsilon \bar{x}_N^R(0) = x_T + O(\epsilon^{N+1})$ by construction,

$$x^*(\bar{T}_N + \epsilon^N \gamma^*) = -\epsilon^{-N+1} \bar{x}_N^L\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) - \frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\bar{x}}_N^O(s) ds + O(\epsilon). \quad (4.22)$$

Similarly,

$$z^*(0) = -\epsilon^{-N} \bar{z}_N^R\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) \quad (4.23)$$

$$q^*(\bar{T}_N + \epsilon^N \gamma^*) = -\epsilon^{-N} \bar{q}_N^L\left(\frac{\bar{T}_N + \epsilon^N \gamma^*}{\epsilon}\right) - \frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\bar{q}}_N^O(s) ds + O(\epsilon) \quad (4.24)$$

The transversality condition becomes

$$0 = \phi_t(t) + H(\bar{y}_N(t) + \epsilon^N y^*(t), \bar{w}_N(t) + \epsilon^N w^*(t)) \Big|_{t=T_N + \epsilon^N \gamma^*}. \quad (4.25)$$

We also rewrite (4.25) into linear parts and the remaining nonlinear parts.

$$C_1^T y_1^*(\bar{T}_N) + C_2^T y_2^*(\bar{T}_N) + C_3 \gamma^* = -H^*(y^*, w^*, \gamma^*, \epsilon) \quad (4.26)$$

where

$$\begin{aligned} C_1 &= H_{y_1}(\bar{y}_N^O(\bar{T}_N), \bar{w}_N^O(\bar{T}_N)) \\ C_2 &= H_{y_2}(\bar{y}_N^O(\bar{T}_N), \bar{w}_N^O(\bar{T}_N)) \\ C_3 &= \phi_{tt}(\bar{T}_N) + C_1^T \dot{\bar{y}}_1^O(\bar{T}_N) + C_2^T \dot{\bar{y}}_2^O(\bar{T}_N) \\ \epsilon^N H^*(y^*, w^*, \gamma^*, \epsilon) &= \phi_t(t) + H(\bar{y}_N(t) + \epsilon^N y^*(t), \bar{w}_N(t) + \epsilon^N w^*(t)) \Big|_{t=T_N + \epsilon^N \gamma^*} \\ &\quad - \epsilon^N C_1^T y_1^*(\bar{T}_N) - \epsilon^N C_2^T y_2^*(\bar{T}_N) - \epsilon^N C_3 \gamma^*. \end{aligned} \quad (4.27)$$

In order to obtain some necessary properties for $H^*(y^*, w^*, \gamma^*, \epsilon)$, we need the following lemma.

Lemma 4.1. For the partial sums (4.4) and (4.5) constructed in Chapter III,

$$\begin{aligned} &\phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N) + \epsilon \bar{y}_N^L(\frac{\bar{T}_N}{\epsilon}) + \epsilon \bar{y}_N^R(0), \bar{w}_N^O(\bar{T}_N) + \bar{w}_N^L(\frac{\bar{T}_N}{\epsilon}) + \bar{w}_N^R(0)) \\ &= O(\epsilon^{N+1}). \end{aligned}$$

Proof. Since $\bar{y}_N^L(\frac{\bar{T}_N}{\epsilon})$, $\bar{w}_N^L(\frac{\bar{T}_N}{\epsilon})$ are $O(\epsilon^{N+1})$,

$$\begin{aligned} &\phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N) + \epsilon \bar{y}_N^L(\frac{\bar{T}_N}{\epsilon}) + \epsilon \bar{y}_N^R(0), \bar{w}_N^O(\bar{T}_N) + \bar{w}_N^L(\frac{\bar{T}_N}{\epsilon}) + \bar{w}_N^R(0)) \\ &= \phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N) + \epsilon \bar{y}_N^R(0), \bar{w}_N^O(\bar{T}_N) + \bar{w}_N^R(0)) + O(\epsilon^{N+1}). \end{aligned} \quad (4.28)$$

Now consider a function $V(\sigma)$ defined by

$$\begin{aligned} V(\sigma) &= H(\bar{y}_N^O(\bar{T}_N - \epsilon\sigma) + \epsilon\bar{y}_N^R(\sigma), \bar{w}_N^O(\bar{T}_N - \epsilon\sigma) + \bar{w}_N^R(\sigma)) \\ &\quad - H(\bar{y}_N^O(\bar{T}_N - \epsilon\sigma), \bar{w}_N^O(\bar{T}_N - \epsilon\sigma)). \end{aligned}$$

Since $\bar{y}_N^R(\frac{\bar{T}_N}{\epsilon})$ and $\bar{w}_N^R(\frac{\bar{T}_N}{\epsilon})$ are $O(\epsilon^{N+1})$,

$$\begin{aligned} V(\frac{\bar{T}_N}{\epsilon}) &= H(\bar{y}_N^O(0) + \epsilon\bar{y}_N^R(\frac{\bar{T}_N}{\epsilon}), \bar{w}_N^O(0) + \bar{w}_N^R(\frac{\bar{T}_N}{\epsilon})) - H(\bar{y}_N^O(0), \bar{w}_N^O(0)) \\ &= O(\epsilon^{N+1}). \end{aligned}$$

Using (4.9) (4.10) and (4.13) (4.14), differentiation of $V(\sigma)$ with respect to σ gives

$$\frac{dV(\sigma)}{d\sigma} = O(\epsilon^{N+1}\delta e^{-\nu\sigma}).$$

Hence, for some K ,

$$\left| \frac{dV(\sigma)}{d\sigma} \right| \leq K\epsilon^{N+1}\delta e^{-\nu\sigma}.$$

By integrating,

$$\begin{aligned} |V(0)| &\leq \left| V(\frac{\bar{T}_N}{\epsilon}) \right| + \left| \int_{\frac{\bar{T}_N}{\epsilon}}^0 \frac{dV(\sigma)}{d\sigma} d\sigma \right| \\ &= O(\epsilon^{N+1}). \end{aligned}$$

From the construction of the outer expansion (3.18),

$$\phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N), \bar{w}_N^O(\bar{T}_N)) = O(\epsilon^{N+1}).$$

Hence,

$$\begin{aligned} &\phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N) + \epsilon\bar{y}_N^R(0), \bar{w}_N^O(\bar{T}_N) + \bar{w}_N^R(0)) \\ &= \phi_t(\bar{T}_N) + H(\bar{y}_N^O(\bar{T}_N), \bar{w}_N^O(\bar{T}_N)) + V(0) \\ &= O(\epsilon^{N+1}). \end{aligned} \tag{4.29}$$

By combining (4.29) with (4.28), the proof is complete. ■

For the nonlinear parts of (4.17) and (4.18) which will be considered as forcing terms, we state some properties to be useful in proving our main result.

Lemma 4.2. Let $\psi = e^{-\nu t/\epsilon} + e^{-\nu(T_N - \epsilon^{N+1}K - t)/\epsilon}$. Given $K > 0$, $u^*(y^*, w^*, \gamma^*, \epsilon)$, $U^*(y^*, w^*, \gamma^*, \epsilon)$ and $H^*(y^*, w^*, \gamma^*, \epsilon)$ have the following properties.

1. There exist a $k_1 > 0$ and $\epsilon_1 > 0$ such that when $|y^*|, |w^*|, |\gamma^*| \leq K\epsilon$ for $0 < \epsilon \leq \epsilon_1$

$$1) \quad |u^*(y^*, w^*, \gamma^*, \epsilon)| \leq k_1 \left(\epsilon + \delta\psi + (\epsilon + \delta_0\psi)(|y^*| + |w^*|) + (\epsilon + \delta_0)\frac{\psi}{\epsilon}|\gamma^*| \right)$$

$$2) \quad |U^*(y^*, w^*, \gamma^*, \epsilon)| \leq k_1 (\epsilon + (\epsilon + \delta_0)(|y^*| + |w^*| + |\gamma^*|))$$

$$3) \quad |H^*(y^*, w^*, \gamma^*, \epsilon)| \leq k_1 (\epsilon + (\epsilon + \delta_0)(|y^*| + |w^*| + |\gamma^*|)).$$

2. For each $k_2 \geq 0$, there exists a $\lambda_1(k_2)$ such that $|\hat{y}|, |\tilde{y}|, |\hat{w}|, |\tilde{w}|, |\hat{\gamma}|, |\tilde{\gamma}| \leq K\epsilon$, $0 \leq \delta_0 \leq \lambda_1(k_2)$ and $0 < \epsilon \leq \lambda_1(k_2)$ imply

$$1) \quad |u^*(\hat{y}, \hat{w}, \hat{\gamma}, \epsilon) - u^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \leq k_2 \left(|\hat{y} - \tilde{y}| + |\hat{w} - \tilde{w}| + \frac{\psi}{\epsilon}|\hat{\gamma} - \tilde{\gamma}| \right)$$

$$2) \quad |U^*(\hat{y}, \hat{w}, \hat{\gamma}, \epsilon) - U^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \leq k_2 (|\hat{y} - \tilde{y}| + |\hat{w} - \tilde{w}| + |\hat{\gamma} - \tilde{\gamma}|)$$

$$3) \quad |H^*(\hat{y}, \hat{w}, \hat{\gamma}, \epsilon) - H^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \leq k_2 (|\hat{y} - \tilde{y}| + |\hat{w} - \tilde{w}| + |\hat{\gamma} - \tilde{\gamma}|).$$

Proof. Using $O(e^{-\nu t/\epsilon}) = O(\epsilon^k)$ for $t \geq \bar{T}_N/2$ and $O(e^{-\nu(T_N - \epsilon^{N+1}K - t)/\epsilon}) = O(\epsilon^k)$ for $t \leq \bar{T}_N/2$ for an arbitrary positive k , one can show that in a ball $|y^*|, |w^*|, |\gamma^*| \leq K\epsilon$,

$$u^*(0, 0, 0, \epsilon) = O(\epsilon + \delta\psi) \quad U^*(0, 0, 0, \epsilon) = O(\epsilon)$$

$$u_{y^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0\psi) \quad u_{w^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0\psi)$$

$$U_{y^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0\psi) \quad U_{w^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0\psi)$$

$$u_{\gamma^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0)O\left(\frac{\psi}{\epsilon}\right) \quad U_{\gamma^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon\psi + \delta_0\psi).$$

Hence, properties 1.1), 1.2), 2.1) and 2.2) directly follow from the mean value theorem.

Since $H_{\gamma^*}^*(y^*, w^*, \gamma^*, \epsilon) = O(\epsilon + \delta_0)$ from (4.27),

$$|H^*(\hat{y}, \hat{w}, \hat{\gamma}, \epsilon) - H^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon)| \leq k(\epsilon + \delta_0)|\hat{\gamma} - \tilde{\gamma}| \quad (4.30)$$

for some $k > 0$.

For simplicity, we define

$$H_y^f(t) = H_y^T(\bar{y}_N(t) + \epsilon^N y^*(t), \bar{w}_N(t) + \epsilon^N w^*(t)) \quad H_y^O = H_y^T(y_N^O(\bar{T}_N), w_N^O(\bar{T}_N))$$

$$\hat{H}_y^f = H_y^T(\bar{y}_N(t) + \epsilon^N \hat{y}(t), \bar{w}_N(t) + \epsilon^N \hat{w}(t)),$$

and $\hat{H}_w^f$, $\tilde{H}_y^f$ and $\tilde{H}_w^f$ are also similarly defined.

In order to estimate $|H^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon) - H^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)|$, we rewrite (4.27) into the following form,

$$H^* = H_1^* + H_2^*$$

where

$$\begin{aligned} H_1^* &= H(\bar{y}_N(\bar{T}_N) + \epsilon^N y^*(\bar{T}_N), \bar{w}_N(\bar{T}_N) + \epsilon^N w^*(\bar{T}_N)) - \epsilon^N H_y^O y^*(\bar{T}_N) \\ &\quad + \phi_t(\bar{T}_N + \epsilon^N \gamma^*) - \epsilon^N \phi_{tt}(\bar{T}_N) \gamma^* \\ H_2^* &= \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \left(H_y^f(\dot{\bar{y}}_N + \epsilon^N \dot{y}^*) + H_w^f(\dot{\bar{w}}_N + \epsilon^N \dot{w}^*) - H_y^O \dot{y}^O(\bar{T}_N) \right) ds. \end{aligned}$$

Since

$$H_{1y^*}^* = H_y^f(\bar{T}_N) - H_y^O = O(\epsilon + \delta_0)$$

$$H_{1w^*}^* = H_w^f(\bar{T}_N)$$

$$\begin{aligned}
&= H_w(y_N^O(\bar{T}_N), w_N^O(\bar{T}_N)) + O(\epsilon + \delta_0) \\
&= O(\epsilon + \delta_0) \text{ from (4.10),}
\end{aligned}$$

one can deduce that for some $k > 0$,

$$|H_1^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon) - H_1^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \leq k(\epsilon + \delta_0)(|\hat{y} - \tilde{y}| + |\hat{w} - \tilde{w}|). \quad (4.31)$$

$|H_2^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon) - H_2^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)|$ can be expressed by

$$\begin{aligned}
&|H_2^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon) - H_2^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \\
&\leq \int_{T_N}^{T_N + \epsilon^N \tilde{\gamma}} |\hat{H}_y^f - \tilde{H}_y^f| |\dot{\hat{y}}_N + \epsilon^N \dot{\hat{y}}| + \epsilon^N |\tilde{H}_y^f| |\dot{\hat{y}} - \dot{\tilde{y}}| ds \\
&\quad + \int_{T_N}^{T_N + \epsilon^N \tilde{\gamma}} |\hat{H}_w^f - \tilde{H}_w^f| |\dot{\hat{w}}_N + \epsilon^N \dot{\hat{w}}| + \epsilon^N |\tilde{H}_w^f| |\dot{\hat{w}} - \dot{\tilde{w}}| ds. \quad (4.32)
\end{aligned}$$

We use the fact that for some $k > 0$,

$$\begin{aligned}
\dot{\hat{y}}_N + \epsilon^N \dot{\hat{y}} &= O(1) & \tilde{H}_y^f &= O(1) \\
\dot{\hat{w}}_N + \epsilon^N \dot{\hat{w}} &= O\left(\frac{\epsilon + \delta_0 \psi}{\epsilon}\right) & \tilde{H}_w^f &= O(\epsilon + \delta_0 \psi)
\end{aligned}$$

and

$$\begin{aligned}
|\hat{H}_y^f - \tilde{H}_y^f| &\leq k|\hat{y} - \tilde{y}| & |\dot{\hat{y}} - \dot{\tilde{y}}| &\leq k|\hat{y} - \tilde{y}| \\
|\hat{H}_w^f - \tilde{H}_w^f| &\leq k|\hat{w} - \tilde{w}| & |\dot{\hat{w}} - \dot{\tilde{w}}| &\leq \frac{k}{\epsilon}|\hat{w} - \tilde{w}|.
\end{aligned}$$

Hence, equation (4.32) becomes

$$|H_2^*(\hat{y}, \hat{w}, \tilde{\gamma}, \epsilon) - H_2^*(\tilde{y}, \tilde{w}, \tilde{\gamma}, \epsilon)| \leq k(\epsilon + \delta_0)(|\hat{y} - \tilde{y}| + |\hat{w} - \tilde{w}|) \quad (4.33)$$

for some $k > 0$.

By combining (4.30) and (4.31) with (4.33), property 2.3) is established. By using (4.27) and Lemma 4.1,

$$H^*(0, 0, 0, \epsilon) = \frac{1}{\epsilon^N} \left(\phi_t(\bar{T}_N) + H(\bar{y}_N, \bar{w}_N) \right) = O(\epsilon).$$

Hence,

$$\begin{aligned} |H^*(y^*, w^*, \gamma^*, \epsilon)| &\leq |H^*(0, 0, 0, \epsilon)| + |H^*(y^*, w^*, \gamma^*, \epsilon) - H^*(0, 0, 0, \epsilon)| \\ &\leq k_1 (\epsilon + (\epsilon + \delta_0)(|y^*| + |w^*| + |\gamma^*|)). \end{aligned}$$

This completes the proof. ■

4.2 Transformation of the TPBVP

Now, with the aid of Lemma 4.3 taken from P. Habets's work [27], we will transform the system (4.17) and (4.18) into a tractable form such that the slow variables and the fast variables are weakly coupled.

Lemma 4.3. Suppose that i) $F_1(t)$, $F_2(t)$, $G_1(t)$ and $G_2(t)$ are continuous and ii) $G_2(t)$ has m eigenvalues with real parts $\leq -2\nu < 0$ and m eigenvalues with real part $\geq 2\nu > 0$ for each $t \in [0, \hat{T}]$, where $F_1(t)$, $F_2(t)$, $G_1(t)$ and $G_2(t)$ are $2n \times 2n$, $2n \times 2m$, $2m \times 2n$ and $2m \times 2m$ dimensional matrices, respectively. There exists $\epsilon_2 > 0$ such that for any $\epsilon \in [0, \epsilon_2]$ the equations

$$\epsilon \dot{R}(t) = G_2(t)R(t) - \epsilon R(t)F_1(t) + \epsilon R(t)F_2(t)R(t) - G_1(t)$$

$$\epsilon \dot{S}(t) = \epsilon (F_1(t) - F_2(t)R(t))S(t) - S(t)(G_2(t) + \epsilon R(t)F_2(t)) - F_2(t)$$

have solutions

$$R = R(t, \epsilon) = G_2^{-1}G_1 + O(\epsilon + \hat{\psi}) \quad \text{where} \quad \hat{\psi} = e^{-\nu t/\epsilon} + e^{-\nu(\hat{T}-t)/\epsilon}$$

$$S = S(t, \epsilon) = O(1).$$

Let

$$F_1 = u_y \quad F_2 = u_w \quad G_1 = U_y \quad G_2 = U_w$$

Since the conditions for Lemma 4.3 are satisfied, $R(t)$ and $S(t)$ have solutions

$$R(t) = U_w^{-1}U_y + O(\epsilon + \hat{\psi}) \quad S(t) = O(1).$$

Consider the change of variables,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} I + \epsilon SR & \epsilon S \\ M^{-1}R & M^{-1} \end{bmatrix} \begin{bmatrix} y^* \\ w^* \end{bmatrix} \quad (4.34)$$

$$\begin{bmatrix} y^* \\ w^* \end{bmatrix} = \begin{bmatrix} I & -\epsilon SM \\ -R & M + \epsilon SRM \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad (4.35)$$

where $M(t)$ is defined in Assumption A.5.

Lemma 4.4. With the change of variables (4.34), the TPBVP (4.17) and (4.18) can be transformed into

$$\dot{\alpha} = A(t)\alpha + l^*(\alpha, \beta, \gamma^*, \epsilon) \quad (4.36)$$

$$\epsilon \dot{\beta} = B(t)\beta + m^*(\alpha, \beta, \gamma^*, \epsilon) \quad (4.37)$$

where

$$A(t) = u_y - u_w U_w^{-1} U_y$$

$$B(t) = M^{-1} U_w M = \begin{bmatrix} B_1(t) & 0 \\ 0 & -B_1^T(t) \end{bmatrix}$$

Moreover, $l^*(\alpha, \beta, \gamma^*, \epsilon)$, $m^*(\alpha, \beta, \gamma^*, \epsilon)$ and $H^*(\alpha, \beta, \gamma^*, \epsilon)$ have the same properties as in Lemma 4.2. Thus,

1. There exist a $k_1 > 0$ and $\epsilon_1 > 0$ such that when $|\alpha|, |\beta|, |\gamma^*| \leq K\epsilon$ for $0 < \epsilon \leq \epsilon_1$,

$$\begin{aligned} 1) \quad & |l^*(\alpha, \beta, \gamma^*, \epsilon)| \leq k_1 \left(\epsilon + \delta\psi + (\epsilon + \delta_0\psi)(|\alpha| + |\beta|) + (\epsilon + \delta_0)\frac{\psi}{\epsilon}|\gamma^*| \right) \\ 2) \quad & |m^*(\alpha, \beta, \gamma^*, \epsilon)| \leq k_1 (\epsilon + (\epsilon + \delta_0)(|\alpha| + |\beta| + |\gamma^*|)) \\ 3) \quad & |H^*(\alpha, \beta, \gamma^*, \epsilon)| \leq k_1 (\epsilon + (\epsilon + \delta_0)(|\alpha| + |\beta| + |\gamma^*|)). \end{aligned}$$

2. For each $k_2 \geq 0$, there exists a $\lambda_1(k_2)$ such that $|\hat{\alpha}|, |\tilde{\alpha}|, |\hat{\beta}|, |\tilde{\beta}|, |\hat{\gamma}|, |\tilde{\gamma}| \leq K\epsilon$, $0 \leq \delta_0 \leq \lambda_1(k_2)$ and $0 < \epsilon \leq \lambda_1(k_2)$ imply

$$\begin{aligned} 1) \quad & |l^*(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \epsilon) - l^*(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \epsilon)| \leq k_2 \left(|\hat{\alpha} - \tilde{\alpha}| + |\hat{\beta} - \tilde{\beta}| + \frac{\psi}{\epsilon}|\hat{\gamma} - \tilde{\gamma}| \right) \\ 2) \quad & |m^*(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \epsilon) - m^*(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \epsilon)| \leq k_2 (|\hat{\alpha} - \tilde{\alpha}| + |\hat{\beta} - \tilde{\beta}| + |\hat{\gamma} - \tilde{\gamma}|) \\ 3) \quad & |H^*(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \epsilon) - H^*(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \epsilon)| \leq k_2 (|\hat{\alpha} - \tilde{\alpha}| + |\hat{\beta} - \tilde{\beta}| + |\hat{\gamma} - \tilde{\gamma}|). \end{aligned}$$

Proof. By substituting (4.34) into (4.17) and (4.18), one can obtain (4.36) and (4.37). Also, $l^*(\alpha, \beta, \gamma^*, \epsilon)$ and $m^*(\alpha, \beta, \gamma^*, \epsilon)$ are expressed by

$$l^*(\alpha, \beta, \gamma^*, \epsilon) = \hat{A}\alpha + \tilde{A}\beta + \tilde{u}^*$$

$$m^*(\alpha, \beta, \gamma^*, \epsilon) = \tilde{B}\beta + \tilde{U}^*$$

where

$$\hat{A} = u_w (U_w^{-1} U_y - R) = O(\epsilon + \hat{\psi})$$

$$\begin{aligned} \tilde{A} &= \epsilon^2 (u_y SSRM - u_w RSSRM - u_y S RSM + u_w RSRSM) \\ &= O(\epsilon^2) \end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \tilde{u}^* \\ \tilde{U}^* \end{bmatrix} &= \begin{bmatrix} I + \epsilon SR & S \\ \epsilon M^{-1}R & M^{-1} \end{bmatrix} \begin{bmatrix} u^* \\ U^* \end{bmatrix} \\
\tilde{B} &= \epsilon \left(M^{-1}Ru_wM + M^{-1}U_wSRM - M^{-1}\dot{M}M^{-1}U_wRSM \right) \\
&\quad + \epsilon^2 \left(M^{-1}Ru_wSRM + M^{-1}\dot{M}M^{-1}RSM \right. \\
&\quad \left. - M^{-1}\dot{M}M^{-1}SRM - M^{-1}Ru_wRSM \right) = O(\epsilon).
\end{aligned}$$

From (4.34), for some $K > 0$,

$$\begin{aligned}
|\alpha| &\leq |y^*| + \epsilon K(|y^*| + |w^*|) \\
|\beta| &\leq K(|y^*| + |w^*|).
\end{aligned}$$

Hence, the properties of $I^*(\alpha, \beta, \gamma^*, \epsilon)$, $m^*(\alpha, \beta, \gamma^*, \epsilon)$ and $H^*(\alpha, \beta, \gamma^*, \epsilon)$ follow directly from the definition of $I^*(\alpha, \beta, \epsilon)$, $m^*(\alpha, \beta, \epsilon)$ and $H^*(\alpha, \beta, \gamma^*, \epsilon)$. ■

We remark that $H^*(\alpha, \beta, \gamma^*, \epsilon)$ in Lemma 4.4 is obtained by substituting (4.34) into $H^*(y^*, w^*, \gamma^*, \epsilon)$.

In terms of the new variables, the boundary conditions and the transversality condition will be derived. Let

$$P_1 = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 \\ I_{n \times n} & 0 \end{bmatrix} \quad Q = \begin{bmatrix} I_{m \times m} & 0 \\ 0 & 0 \end{bmatrix}.$$

By using (4.35), the boundary conditions (4.21) – (4.24) and the transversality condition (4.26) become

$$\begin{aligned}
\begin{bmatrix} \alpha_1(0) \\ \alpha_1(\bar{T}_N + \epsilon^N \gamma^*) \end{bmatrix} &= \begin{bmatrix} 0 \\ -\frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\tilde{x}}_N^O(s) ds \end{bmatrix} + \epsilon P_1 S M \beta(0) \\
&\quad + \epsilon P_2 S M \beta(\bar{T}_N + \epsilon^N \gamma^*) + O(e^{-\nu(\bar{T}_N + \epsilon^N \gamma^*)/\epsilon})
\end{aligned} \tag{4.38}$$

$$\begin{aligned} \bar{M}(\gamma^*, \epsilon) \begin{bmatrix} \beta_1(0) \\ \beta_2(\bar{T}_N + \epsilon^N \gamma^*) \end{bmatrix} &= Q R \alpha(0) + (I - Q) R \alpha(\bar{T}_N + \epsilon^N \gamma^*) \\ &- \begin{bmatrix} 0 \\ \frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{q}_N^O(s) ds \end{bmatrix} + O(e^{-\nu(\bar{T}_N + \epsilon^N \gamma^*)/\epsilon}) \end{aligned} \quad (4.39)$$

$$\begin{aligned} C_1^T \alpha_1(\bar{T}_N) + C_2^T \alpha_2(\bar{T}_N) + C_3 \gamma^* &= -H^*(\alpha, \beta, \gamma^*, \epsilon) \\ &+ \epsilon \begin{bmatrix} C_1^T & C_2^T \end{bmatrix} (P_1 S M + P_2 S M) \beta(\bar{T}_N) \end{aligned} \quad (4.40)$$

where

$$\bar{M}(\gamma^*, \epsilon) = Q(I + \epsilon S R)M(0) + (I - Q)(I + \epsilon S R)M(\bar{T}_N + \epsilon^N \gamma^*). \quad (4.41)$$

Consider the linear parts of (4.36) and (4.37). Let $\Theta_1(t, s)$ and $\Theta_2(t, s)$ be the fundamental matrices satisfying the following matrix differential equations

$$\begin{aligned} \dot{\Theta}_1(t, s) &= \frac{1}{\epsilon} B_1(t) \Theta_1(t, s) \quad \Theta_1(s, s) = I_{m \times m} \\ \dot{\Theta}_2(t, s) &= -\frac{1}{\epsilon} B_1^T(t) \Theta_2(t, s) \quad \Theta_2(s, s) = I_{m \times m}. \end{aligned}$$

It was shown by Flatto and Levinson [28] that $\Theta_1(t, s)$ and $\Theta_2(t, s)$ have the following bound if $B_1(t)$ has m eigenvalues with real parts $\leq -2\nu < 0$ for each $t \in [0, \hat{T}]$.

$$|\Theta_1(t, s)| \leq k_3 e^{-\nu(t-s)/\epsilon} \quad 0 \leq s \leq t \leq \hat{T} \quad (4.42)$$

$$|\Theta_2(t, s)| \leq k_3 e^{-\nu(s-t)/\epsilon} \quad 0 \leq t \leq s \leq \hat{T} \quad (4.43)$$

where k_3 is a positive constant independent of ϵ .

Let $\Phi(t, 0)$ be the fundamental matrix such that

$$\dot{\Phi}(t, s) = A(t) \Phi(t, s) \quad \Phi(s, s) = I_{2n \times 2n}.$$

Then, $|\Phi(t, s)|$ is bounded by

$$|\Phi(t, s)| \leq k_4 e^{k_5 |t-s|} \quad (4.44)$$

for some constant k_4 and k_5 [29].

Using the transition matrix $\Phi(t, s)$ of the slow system, we rewrite (4.38) and (4.40) into a convenient form. Since

$$\begin{aligned} \alpha_1(\bar{T}_N + \epsilon^N \gamma^*) &= \alpha_1(\bar{T}_N) + \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\alpha}_1(s) ds \\ &= \Phi_{11}(\bar{T}_N, 0) \alpha_1(0) + \Phi_{12}(\bar{T}_N, 0) \alpha_2(0) \\ &\quad + \int_0^{\bar{T}_N} (\Phi_{11}(\bar{T}_N, s) l_1^* + \Phi_{12}(\bar{T}_N, s) l_2^*) ds \\ &\quad + \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} \dot{\alpha}_1(s) ds \\ \beta(\bar{T}_N + \epsilon^N \gamma^*) &= \beta(\bar{T}_N) + \frac{1}{\epsilon} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (B\beta + m^*) ds, \end{aligned}$$

equation (4.38) can be rewritten by

$$\begin{aligned} &\begin{bmatrix} \alpha_1(0) \\ \Phi_{11}(\bar{T}_N, 0) \alpha_1(0) + \Phi_{12}(\bar{T}_N, 0) \alpha_2(0) + \dot{x}_N^O(\bar{T}_N) \gamma^* \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (\dot{x}_N^O(s) - \dot{x}_N^O(\bar{T}_N)) ds \end{bmatrix} + \epsilon P_1 S M \beta(0) \\ &\quad + \epsilon P_2 S M \left(\beta(\bar{T}_N) + \frac{1}{\epsilon} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (B\beta + m^*) ds \right) - P_2 \int_0^{\bar{T}_N} \Phi(\bar{T}_N, s) l^* ds \\ &\quad - P_2 \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (A\alpha + l^*) ds + O(\epsilon^{-\nu(\bar{T}_N + \epsilon^N \gamma^*)/\epsilon}). \end{aligned} \quad (4.45)$$

Hence, combining (4.45) with (4.40) gives

$$L(\bar{T}_N, \epsilon) \begin{bmatrix} \alpha_1(0) \\ \alpha_2(0) \\ \gamma^* \end{bmatrix} = \begin{bmatrix} h_1(\alpha, \beta, \gamma^*) \\ h_2(\alpha, \beta, \gamma^*) \\ h_3(\alpha, \beta, \gamma^*) \end{bmatrix} \quad (4.46)$$

where

$$L(\bar{T}_N, \epsilon) = \begin{bmatrix} I & 0 & 0 \\ \Phi_{11}(\bar{T}_N, 0) & \Phi_{12}(\bar{T}_N, 0) & \dot{x}_N^O(\bar{T}_N) \\ L_{31} & L_{32} & C_3 \end{bmatrix} \quad (4.47)$$

$$L_{31} = C_1^T \Phi_{11}(\bar{T}_N, 0) + C_2^T \Phi_{21}(\bar{T}_N, 0)$$

$$L_{32} = C_1^T \Phi_{12}(\bar{T}_N, 0) + C_2^T \Phi_{22}(\bar{T}_N, 0)$$

$$\begin{bmatrix} h_1(\alpha, \beta, \gamma^*) \\ h_2(\alpha, \beta, \gamma^*) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (\dot{x}_N^O(s) - \dot{x}_N^O(\bar{T}_N)) ds \\ + \epsilon P_1 S M \beta(0) - P_2 \int_0^{\bar{T}_N} \Phi(\bar{T}_N, s) l^* ds \\ + \epsilon P_2 S M \left(\beta(\bar{T}_N) + \frac{1}{\epsilon} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (B\beta + m^*) ds \right) \\ - P_2 \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \gamma^*} (A\alpha + l^*) ds + O(e^{-\nu(\bar{T}_N + \epsilon^N \gamma^*)/\epsilon}) \end{bmatrix} \quad (4.48)$$

$$\begin{aligned} h_3(\alpha, \beta, \gamma^*) &= -H^* + \epsilon \begin{bmatrix} C_1^T & C_2^T \end{bmatrix} (P_1 S M + P_2 S M) \beta(\bar{T}_N) \\ &\quad - \begin{bmatrix} C_1^T & C_2^T \end{bmatrix} \int_0^{\bar{T}_N} (P_1 + P_2) \Phi(\bar{T}_N, s) l^* ds. \end{aligned} \quad (4.49)$$

Lemma 4.5. There exists $\epsilon_3 > 0$ such that for all $0 \leq \epsilon \leq \epsilon_3$, i) the $(2n+1) \times (2n+1)$ matrix $L(\bar{T}_N, \epsilon)$ defined in (4.46) has a bounded inverse and ii) given $K > 0$, the $2m \times 2m$ matrix $\bar{M}(\gamma^*, \epsilon)$ defined in (4.41) has a bounded inverse if $|\gamma^*| \leq K\epsilon$.

Proof. Assumption A.3 implies that the linear homogeneous variational TPBVP

$$\dot{x}_0^O = A_{11}x_0^O + A_{12}p_0^O \quad (4.50)$$

$$\dot{p}_0^O = A_{21}x_0^O + A_{22}p_0^O \quad (4.51)$$

with the trivial boundary conditions

$$x_0^O(0) = 0 \quad (4.52)$$

$$x_0^O(T_0) + \dot{x}_0^O(T_0)T_1 = 0 \quad (4.53)$$

$$\begin{aligned} & H_x^T x_0^O(T_0) + H_p^T p_0^O(T_0) + \left(\phi_{tt}(T_0) + H_x^T \dot{x}_0^O(T_0) + H_p^T \dot{p}_0^O(T_0) \right) T_1 \\ & = 0 \end{aligned} \quad (4.54)$$

has only the trivial solution, where

$$A(t) = u_y - u_w U_w^{-1} U_y.$$

Since

$$x_0^O(T_0) = \Phi_{11}(T_0, 0)x_0^O(T_0) + \Phi_{12}(T_0, 0)p_0^O(T_0)$$

$$p_0^O(T_0) = \Phi_{21}(T_0, 0)x_0^O(T_0) + \Phi_{22}(T_0, 0)p_0^O(T_0),$$

the boundary conditions (4.52) – (4.54) are equivalent to

$$L(T_0, 0) \begin{bmatrix} x_0^O(0) \\ p_0^O(0) \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (4.55)$$

Suppose that $L(T_0, 0)$ is not invertible. Then, any non-trivial solution starting with a non-zero initial condition satisfying (4.55) is a solution of the TPBVP

(4.50) – (4.54). Hence, we proved that $L(T_0, 0)$ is invertible by contradiction. Since all the entries in $L(\bar{T}_N, \epsilon)$ are continuous functions of ϵ , there exists an $\epsilon_3 > 0$ such that $L(\bar{T}_N, \epsilon)$ has a bounded inverse. This completes the first part of the proof.

From (4.41), $\bar{M}(\gamma^*, \epsilon)$ can be written by

$$\bar{M}(\gamma^*, \epsilon) = \begin{bmatrix} M_{11}(0) & 0 \\ 0 & M_{22}(\bar{T}_N + \epsilon^N \gamma^*) \end{bmatrix} + O(\epsilon).$$

Since $M_{11}(0)$ and $M_{22}(T_0)$ are invertible from Assumption A.5, $\bar{M}(\gamma^*, \epsilon)$ is invertible for sufficiently small ϵ . ■

4.3 Main Result

Let $\mathcal{C}(I, \mathcal{R}^n)$ be a Banach space of continuous functions on a compact interval I with values in $\mathcal{R}^n$ endowed with a norm

$$|x(t)|_\infty = \sup_{t \in I} |x(t)|$$

where $I = [0, \bar{T}_N + \epsilon^{N+1}K]$ for some positive K and ϵ . We define a Banach space $\mathcal{W}$ with a norm $|w|_e$ by

$$\mathcal{W} = \mathcal{C}(I, \mathcal{R}^{2n}) \times \mathcal{C}(I, \mathcal{R}^{2m}) \times \mathcal{R}$$

$$|w|_e = |w_1(t)|_\infty + |w_2(t)|_\infty + |w_3|$$

where

$$w = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3 \end{bmatrix}$$

$$w_1(t) \in \mathcal{C}(I, \mathcal{R}^{2n}) \quad w_2(t) \in \mathcal{C}(I, \mathcal{R}^{2m}) \quad w_3 \in \mathcal{R}.$$

For an $n \times m$ matrix $A(t)$, $\|A(t)\|$ denotes $\sup_{t \in I} |A(t)|$.

Consider a closed subset $\mathcal{B}$ of $\mathcal{W}$ defined by

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$$

where

$$\mathcal{B}_1 = \{x(t) \in \mathcal{C}(I, \mathcal{R}^{2n}) \mid |x(t)|_\infty \leq K\epsilon\}$$

$$\mathcal{B}_2 = \{x(t) \in \mathcal{C}(I, \mathcal{R}^{2m}) \mid |x(t)|_\infty \leq K\epsilon\}$$

$$\mathcal{B}_3 = \{x \in \mathcal{R} \mid |x| \leq K\epsilon\}.$$

Now, we construct a nonlinear operator Ψ acting on $\mathcal{B}$ such that to solve the TPBVP (4.36) and (4.37) with the boundary condition (4.38) – (4.40) is equivalent to find a fixed point α , β and γ^* of the nonlinear operator Ψ . Thus,

$$\Psi = \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \Gamma_3 \end{bmatrix} : \mathcal{B} \rightarrow \mathcal{C}(I, \mathcal{R}^{2n}) \times \mathcal{C}(I, \mathcal{R}^{2m}) \times \mathcal{R}.$$

For $\pi \in \mathcal{B}_1$, $\xi \in \mathcal{B}_2$ and $\rho \in \mathcal{B}_3$, we define Γ such that

$$\begin{aligned} \Gamma &: \mathcal{B} \rightarrow \mathcal{R}^{2n} \times \mathcal{R}^{2m} \times \mathcal{R} \\ \Gamma &= \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} = \begin{bmatrix} \Gamma_1(\pi, \xi, \rho) \\ \Gamma_2(\pi, \xi, \rho) \\ \Gamma_3(\pi, \xi, \rho) \end{bmatrix} = L(\bar{T}_N, \epsilon)^{-1} \begin{bmatrix} h_1(\pi, \xi, \rho) \\ h_2(\pi, \xi, \rho) \\ h_3(\pi, \xi, \rho) \end{bmatrix}. \end{aligned} \quad (4.56)$$

Since $L(\bar{T}_N, \epsilon)$ has a bounded inverse (Lemma 4.5), we can obtain initial conditions Γ_1 and Γ_2 of the slow variable and a final time Γ_3 from a map (4.56).

Therefore, a trajectory of the slow variable can be determined by $\Psi_1(t)$

$$\Psi_1(t) : \mathcal{B} \rightarrow \mathcal{C}(I, \mathcal{R}^{2n})$$

$$\begin{aligned}
\Psi_1(t) &= \Psi_1(t, \pi, \xi, \rho) \\
&= \Phi(t, 0) \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} + \int_0^t \Phi(t, s) l^*(\pi, \xi, \rho, \epsilon) ds.
\end{aligned} \tag{4.57}$$

Since $\bar{M}(\rho, \epsilon)$ is invertible, a boundary condition for the fast variable can be determined from (4.39).

$$\begin{aligned}
\Lambda &: B \rightarrow \mathcal{R}^{2m} \\
\Lambda &= \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1(\pi, \xi, \rho) \\ \Lambda_2(\pi, \xi, \rho) \end{bmatrix} \\
&= \bar{M}(\Gamma_3, \epsilon)^{-1} \left(Q R \Psi_1(0) + (I - Q) R \Psi_1(\bar{T}_N + \epsilon^N \Gamma_3) \right) \\
&\quad - \bar{M}(\Gamma_3, \epsilon)^{-1} \begin{bmatrix} 0 \\ \frac{1}{\epsilon^N} \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \Gamma_3} \dot{\bar{q}}_N^O(s) ds \end{bmatrix} + O(e^{-\nu(\bar{T}_N + \epsilon^N \Gamma_3)/\epsilon}).
\end{aligned} \tag{4.58}$$

Since a boundary condition for the fast variable is determined, we obtain a trajectory for the fast variable by defining a map $\Psi_2(t)$ as follows.

$$\begin{aligned}
\Psi_2(t) &: B \rightarrow \mathcal{C}(I, \mathcal{R}^{2m}) \\
\Psi_2(t) &= \Psi_2(t, \pi, \xi, \rho) \\
&= \begin{bmatrix} \Theta_1(t, 0) \Lambda_1 \\ \Theta_2(t, \bar{T}_N + \epsilon^N \Gamma_3) \Lambda_2 \end{bmatrix} \\
&\quad + \frac{1}{\epsilon} \begin{bmatrix} \int_0^t \Theta_1(t, s) m_1^*(\pi, \xi, \rho, \epsilon) ds \\ \int_{\bar{T}_N + \epsilon^N \Gamma_3}^t \Theta_2(t, s) m_2^*(\pi, \xi, \rho, \epsilon) ds \end{bmatrix}.
\end{aligned} \tag{4.59}$$

Theorem 4.6. Suppose that Assumptions A.1 – A.5 hold. There exist ϵ^* and δ^* such that when $\epsilon \in (0, \epsilon^*]$ and $|w_0^L(0)|, |w_0^R(0)| \leq \delta^*$, the solution of the TPBPV (4.17) and (4.18) with the boundary condition (4.21) – (4.25) exists and the solution y^* , w^* and γ^* are $O(\epsilon)$.

Proof. It suffices to prove that the nonlinear operator Ψ has a fixed point in a subset of $\mathcal{B}$. Our proof consists of two steps. First, we will show that $|\Psi(t, \pi, \xi, \rho)|_e \leq K\epsilon$. Next, we will show

$$|\Psi(t, \hat{\pi}, \hat{\xi}, \hat{\rho}) - \Psi(t, \tilde{\pi}, \tilde{\xi}, \tilde{\rho})|_e < \left\| \begin{bmatrix} \hat{\pi} - \tilde{\pi} \\ \hat{\xi} - \tilde{\xi} \\ \hat{\rho} - \tilde{\rho} \end{bmatrix} \right\|_e$$

for all

$$\begin{bmatrix} \hat{\pi} \\ \hat{\xi} \\ \hat{\rho} \end{bmatrix}, \quad \begin{bmatrix} \tilde{\pi} \\ \tilde{\xi} \\ \tilde{\rho} \end{bmatrix} \in \mathcal{B}.$$

By a Banach space fixed point theorem [26], this implies that the operator Ψ has a unique fixed point in a subset of $\mathcal{B}$.

Assume that $\hat{T}$ defined in Lemma 4.3 is sufficiently large such that $\hat{T} \geq \bar{T}_N + \epsilon^{N+1}K$. Throughout the proof, for each $i, i = 0, 1, 2, \dots$, c_i denotes a constant which is independent of K and $v(\epsilon)$ denotes a continuous function such that $\lim_{\epsilon \rightarrow 0} v(\epsilon) = 0$.

Recall that $\psi = e^{-\nu t/\epsilon} + e^{-\nu(T_N - \epsilon^{N+1}K - t)/\epsilon}$ in Lemma 4.2. Therefore, for any $t \in I$

$$\int_0^t \psi dt \leq k\epsilon e^{2\nu K\epsilon^N} \text{ for some } k.$$

As a preliminary step, from (4.44)

$$\|\Phi(t, s)\| \leq k_4 e^{k_5(T_N + \epsilon^{N+1}K)}.$$

For any finite $K > 0$, the choice of $\epsilon_4 = 1/K$ gives $\|\Phi(t, s)\| \leq c_0$ for all $0 < \epsilon \leq \epsilon_4$ and $t, s \in I$. Hence, using Lemma 4.4 and $|\pi|, |\xi|, |\rho| \leq K\epsilon$ gives

$$\left| \int_0^t \Phi(t, s) l^*(\pi, \xi, \rho, \epsilon) ds \right|_\infty$$

$$\begin{aligned}
&\leq \int_0^{\bar{T}_N + \epsilon^{N+1}K} \|\Phi(t, s)\| \|l^*(\pi, \xi, \rho, \epsilon)\| ds \\
&\leq c_0 \int_0^{\bar{T}_N + \epsilon^{N+1}K} k_1 (\epsilon + \delta\psi + (\epsilon + \delta_0\psi)(|\pi| + |\xi|) + (\psi + \delta_0\psi/\epsilon)|\rho|) ds \\
&\leq (c_1 + c_2 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon.
\end{aligned} \tag{4.60}$$

Using (4.42) and Lemma 4.4,

$$\begin{aligned}
&\frac{1}{\epsilon} \left| \int_0^t \Theta_1(t, s) m_1^*(\pi, \xi, \rho, \epsilon) ds \right|_\infty \\
&\leq \frac{1}{\epsilon} \left| \int_0^t k_1 k_3 e^{-\nu(t-s)/\epsilon} (\epsilon + (\epsilon + \delta_0)(|\pi| + |\xi| + |\rho|)) ds \right|_\infty \\
&\leq \frac{k_1 k_3}{\nu} (1 + 3K\delta_0 + v(\epsilon)) \epsilon \\
&\leq (c_1 + c_2 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon.
\end{aligned} \tag{4.61}$$

Similarly,

$$\frac{1}{\epsilon} \left| \int_{\bar{T}_N + \epsilon^{N+1}K}^t \Theta_2(t, s) m_2^*(\pi, \xi, \rho, \epsilon) ds \right|_\infty \leq (c_1 + c_2 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon. \tag{4.62}$$

In equations (4.48) and (4.49), since $|\dot{x}_N^O(s) - \dot{x}_N^O(\bar{T}_N)| \leq k|s - \bar{T}_N|$ for some k ,

$$\begin{aligned}
\frac{1}{\epsilon^N} \left| \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \rho} (\dot{x}_N^O(s) - \dot{x}_N^O(\bar{T}_N)) ds \right| &\leq \frac{k}{2} \epsilon^N |\rho|^2 \\
&\leq \frac{k}{2} \epsilon^{N+2} K^2 = v(\epsilon) \epsilon
\end{aligned} \tag{4.63}$$

$$\left| \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \rho} (B|\xi| + m^*) ds \right| \leq v(\epsilon) \epsilon \tag{4.64}$$

$$\left| \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \rho} (A|\pi| + l^*) ds \right| \leq v(\epsilon) \epsilon. \tag{4.65}$$

By using (4.60) and (4.63) – (4.65), one can estimate

$$\left\| \begin{bmatrix} h_1(\pi, \xi, \rho) \\ h_2(\pi, \xi, \rho) \\ h_3(\pi, \xi, \rho) \end{bmatrix} \right\| \leq (c_3 + c_4 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon.$$

Since $L(\bar{T}_N, \epsilon)$ has a bounded inverse (Lemma 4.5), $|\Gamma(\pi, \xi, \rho)|$ is bounded by

$$\begin{aligned} |\Gamma(\pi, \xi, \rho)| &\leq \|L(\bar{T}_N, \epsilon)^{-1}\| \left\| \begin{bmatrix} h_1(\pi, \xi, \rho) \\ h_2(\pi, \xi, \rho) \\ h_3(\pi, \xi, \rho) \end{bmatrix} \right\| \\ &\leq (c_5 + c_6 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon. \end{aligned} \quad (4.66)$$

From a map (4.57), using (4.60) and (4.66) gives

$$\begin{aligned} |\Psi_1(t, \pi, \xi, \rho)|_\infty &\leq \|\Phi(t, 0)\| \left\| \begin{bmatrix} \Gamma_1(\pi, \xi, \rho) \\ \Gamma_2(\pi, \xi, \rho) \end{bmatrix} \right\| + \left| \int_0^t \Phi(t, s) l^*(\pi, \xi, \rho, \epsilon) ds \right|_\infty \\ &\leq (c_7 + c_8 e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon. \end{aligned} \quad (4.67)$$

From (4.58), $|\Lambda(\pi, \xi, \rho)|$ can be written as

$$\begin{aligned} |\Lambda(\pi, \xi, \rho)| &\leq \|\bar{M}(\Gamma_3, \epsilon)^{-1}\| |Q R \Psi_1(0, \pi, \xi, \rho)| \\ &\quad + \|\bar{M}(\Gamma_3, \epsilon)^{-1}\| \|(I - Q) R \Psi_1(\bar{T}_N + \epsilon^N \Gamma_3, \pi, \xi, \rho)\| \\ &\quad + \frac{1}{\epsilon^N} \|\bar{M}(\Gamma_3, \epsilon)^{-1}\| \left| \int_{\bar{T}_N}^{\bar{T}_N + \epsilon^N \Gamma_3(\pi, \xi, \rho, \epsilon)} \dot{\bar{q}}_N^O(s) ds \right| + O(e^{-\nu \bar{T}_N / \epsilon}) \end{aligned}$$

Since $|\dot{\bar{q}}_N^O(t)|$ is a known function which is bounded in the compact interval $[0, \bar{T}_N + \epsilon^{N+1} K]$, using Lemma 4.2 and (4.67) we obtain

$$|\Lambda(\pi, \xi, \rho)| \leq (c_9 + c_{10} e^{2\nu K \epsilon^N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon. \quad (4.68)$$

Finally, from (4.61), (4.62) and (4.68) the map $|\Psi_2(t, \pi, \xi, \rho)|_\infty$ in (4.59) is bounded by

$$|\Psi_2|_\infty \leq \left\| \begin{bmatrix} \Theta_1(t, 0) \Lambda_1(\pi, \xi, \rho) \\ \Theta_2(t, \bar{T}_N + \epsilon^N \Gamma_3) \Lambda_2(\pi, \xi, \rho) \end{bmatrix} \right\|_\infty$$

$$\begin{aligned}
& + \left\| \begin{bmatrix} \frac{1}{\epsilon} \int_0^t \Theta_1(t, s) m_1^*(\pi, \xi, \rho, \epsilon) ds \\ \frac{1}{\epsilon} \int_{T_N + \epsilon N \Gamma_3}^t \Theta_2(t, s) m_2^*(\pi, \xi, \rho, \epsilon) ds \end{bmatrix} \right\|_{\infty} \\
& \leq (c_{11} + c_{12} e^{2\nu K \epsilon N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon.
\end{aligned} \tag{4.69}$$

Let $c_{13} = c_5 + c_7 + c_{11}$ and $c_{14} = c_6 + c_8 + c_{12}$. Then

$$|\Psi(t, \pi, \xi, \rho)|_{\epsilon} \leq (c_{13} + c_{14} e^{\nu K \epsilon N} (\delta_0 K + \delta) + v(\epsilon)) \epsilon$$

We choose sufficiently small ϵ_5 and $\bar{\delta}$ such that

$$c_{13} + c_{14} e^{2\nu K \epsilon N} (\delta_0 K + \delta) + v(\epsilon) \leq K \quad \text{for all } 0 < \epsilon \leq \epsilon_5 \quad 0 \leq \delta_0 \leq \bar{\delta}$$

Consequently,

$$|\Psi(t, \pi, \xi, \rho)|_{\epsilon} \leq K \epsilon$$

This completes the first part of the proof.

Throughout the remainder of the proof, if h is a function of π, ξ and ρ , then $\hat{h}$ and $\tilde{h}$ represent $h(\hat{\pi}, \hat{\xi}, \hat{\rho})$ and $h(\tilde{\pi}, \tilde{\xi}, \tilde{\rho})$, respectively. Also c'_i for $i = 0, 1, 2, \dots$ denotes a constant uniformly bounded in ϵ .

As a basic step, by using Lemma 4.4

$$\begin{aligned}
& \left| \int_0^t \Phi(t, s) (\hat{I}^* - \tilde{I}^*) ds \right|_{\infty} \\
& \leq \|\Phi(t, s)\| \int_0^{T_N + \epsilon^{N+1} K} k_2 (|\hat{\pi} - \tilde{\pi}| + |\hat{\xi} - \tilde{\xi}| + \frac{\psi}{\epsilon} |\hat{\rho} - \tilde{\rho}|) ds \\
& \leq k_2 c'_0 (|\hat{\pi} - \tilde{\pi}|_{\infty} + |\hat{\xi} - \tilde{\xi}|_{\infty} + e^{2\nu K \epsilon N} |\hat{\rho} - \tilde{\rho}|) \\
& \leq k_2 c'_1 (|\hat{\pi} - \tilde{\pi}|_{\infty} + |\hat{\xi} - \tilde{\xi}|_{\infty} + |\hat{\rho} - \tilde{\rho}|) \\
& \frac{1}{\epsilon} \left| \int_0^t \Theta_1(t, s) (\hat{m}_1^* - \tilde{m}_1^*) ds \right|_{\infty} \\
& \leq \frac{1}{\epsilon} \left| \int_0^t k_2 k_3 e^{-\nu(t-s)/\epsilon} (|\hat{\pi} - \tilde{\pi}| + |\hat{\xi} - \tilde{\xi}| + |\hat{\rho} - \tilde{\rho}|) ds \right|_{\infty}
\end{aligned} \tag{4.70}$$

$$\leq k_2 c'_2(|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|). \quad (4.71)$$

Similarly,

$$\frac{1}{\epsilon} \left| \int_{T_N + \epsilon^N \tilde{\Gamma}_3}^t \Theta_2(t, s)(\hat{m}_2^* - \tilde{m}_2^*) ds \right|_\infty \leq k_2 c'_3(|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|). \quad (4.72)$$

By using the basic estimates (4.70) – (4.71), we can obtain

$$|\Gamma(\hat{\pi}, \hat{\xi}, \hat{\rho}) - \Gamma(\tilde{\pi}, \tilde{\xi}, \tilde{\rho})| \leq k_2 c'_4(|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|) \quad (4.73)$$

$$|\Psi_1(t, \hat{\pi}, \hat{\xi}, \hat{\rho}) - \Psi_1(t, \tilde{\pi}, \tilde{\xi}, \tilde{\rho})|_\infty \leq k_2 c'_5(|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|) \quad (4.74)$$

$$|\Lambda(\hat{\pi}, \hat{\xi}, \hat{\rho}) - \Lambda(\tilde{\pi}, \tilde{\xi}, \tilde{\rho})| \leq k_2 c'_6(|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|). \quad (4.75)$$

The details of this computation are long and tedious but straight-forward and so they will be omitted here.

$|\hat{\Psi}_2 - \tilde{\Psi}_2|_\infty$ is obtained as follows. Since from (4.43) $|\Theta_2(t, s)|$ is $O(1)$ for all $t, s \in I$, $|\dot{\Theta}_2(t, s)|$ is $O(1/\epsilon)$ from (4.42). Hence,

$$\begin{aligned} & \|\Theta_2(t, \bar{T}_N + \epsilon^N \hat{\Gamma}_3) - \Theta_2(t, \bar{T}_N + \epsilon^N \tilde{\Gamma}_3)\| \\ & \leq \frac{c'_7}{\epsilon} |\hat{\Gamma} - \tilde{\Gamma}| \leq k_2 \frac{c'_8}{\epsilon} (|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|). \end{aligned} \quad (4.76)$$

Since $\hat{m}_2^*$ is $O(\epsilon)$,

$$\begin{aligned} \frac{1}{\epsilon} \int_{T_N + \epsilon^N \tilde{\Gamma}_3}^{T_N + \epsilon^N \hat{\Gamma}_3} \|\Theta_2(t, s)\| |\hat{m}_2^*| ds & \leq c'_9 |\hat{\Gamma}_3 - \tilde{\Gamma}_3| \\ & \leq k_2 c'_{10} (|\hat{\pi} - \tilde{\pi}|_\infty + |\hat{\xi} - \tilde{\xi}|_\infty + |\hat{\rho} - \tilde{\rho}|). \end{aligned} \quad (4.77)$$

By using (4.71), (4.72) and (4.75) – (4.77), $|\hat{\Psi}_2 - \tilde{\Psi}_2|_\infty$ is bounded by

$$\begin{aligned} |\hat{\Psi}_2 - \tilde{\Psi}_2|_\infty & \leq \|\Theta_1(t, 0)\| |\hat{\Lambda}_1 - \tilde{\Lambda}_1| + \frac{1}{\epsilon} \left| \int_0^t \Theta_1(t, s)(\hat{m}_1^* - \tilde{m}_1^*) ds \right|_\infty \\ & \quad + \|\Theta_2(t, \bar{T}_N + \epsilon^N \hat{\Gamma}_3) - \Theta_2(t, \bar{T}_N + \epsilon^N \tilde{\Gamma}_3)\| |\hat{\Lambda}_2| \end{aligned}$$

$$\begin{aligned}
& + \|\Theta_2(t, \bar{T}_N + \epsilon^N \tilde{\Gamma}_3)\| |\hat{\Lambda}_2 - \tilde{\Lambda}_2| + \frac{1}{\epsilon} \int_{T_N + \epsilon^N \tilde{\Gamma}_3}^{T_N + \epsilon^N \tilde{\Gamma}_3} \|\Theta_2(t, s)\| |\bar{m}_2^*| ds \\
& + \frac{1}{\epsilon} \left| \int_{T_N + \epsilon^N \tilde{\Gamma}_3}^t \Theta_2(t, s) (\hat{m}_2^* - \bar{m}_2^*) ds \right|_{\infty} \\
& \leq k_2 c'_{11} (|\hat{\pi} - \tilde{\pi}|_{\infty} + |\hat{\xi} - \tilde{\xi}|_{\infty} + |\hat{\rho} - \tilde{\rho}|). \tag{4.78}
\end{aligned}$$

Hence, $|\hat{\Psi} - \tilde{\Psi}|_e$ is bounded by

$$|\hat{\Psi} - \tilde{\Psi}|_e \leq k_2 c'_{12} (|\hat{\pi} - \tilde{\pi}|_{\infty} + |\hat{\xi} - \tilde{\xi}|_{\infty} + |\hat{\rho} - \tilde{\rho}|)$$

where $c'_{12} = c'_4 + c'_5 + c'_{11}$.

Now, we choose k_2 such that $k_2 c'_{12} < 1$. Finally, let

$$\epsilon^* = \min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \lambda_1(k_2)) \quad \delta^* = \min(\bar{\delta}, \lambda_1(k_2))$$

Then for $\epsilon \leq \epsilon^*$ and $\delta_0 \leq \delta^*$, the nonlinear operator Ψ has a unique fixed point in a subset of $\mathcal{B}$. This completes the proof. ■

4.4 Concluding Remarks

In the proof of the main result, Assumptions A.1 – A.5 are made on the lower order problem. The fact that these lower order assumptions allow the determination of the asymptotic solution is the utility and power of the singular perturbation approach. We require that the size of the zeroth order boundary layer jumps δ_0 to be sufficiently small. But, this is inevitable in a general nonlinear context. Similar results can be found in Freedman's work [22] which considered a nonlinear free end point optimal control problem with a fixed final time.

In proving our main result, we need three key assumptions; 1) sufficient smoothness of our system, 2) the reduced solution satisfies Jacobi's sufficient condition for a

local minimum and 3) an eigenvalue condition for the fast subsystem. But these assumptions are not strong in our opinion. The smoothness assumption is required to derive an asymptotic series solution up to any degree of accuracy. Jacobi's sufficient condition guarantees a well-posedness of the reduced problem. Thus, without Jacobi's condition, the auxiliary optimization problem is meaningless. The eigenvalue condition for the fast subproblem is inevitable for the application of the singular perturbation method to a control problem since we can not neglect the higher order fast dynamics if the fast dynamics are unstable. But, we only assume that the system matrix of the fast system (in our notation U_w) is not a stable matrix as a time varying system but a pointwise stable matrix.

CHAPTER V NONLINEAR SYSTEMS WITH MULTIPLE PARAMETERS

In this chapter, we consider a nonlinear TPBVP which has multiple parameters. In this case, we can construct an asymptotic solution by a similar method as described in Chapter III in several time scales.

First we consider a nonlinear TPBVP with two small parameters

$$\dot{x} = f(x, y, z) \quad (5.1)$$

$$\epsilon_1 \dot{y} = g(x, y, z) \quad (5.2)$$

$$\epsilon_1 \epsilon_2 \dot{z} = h(x, y, z) \quad (5.3)$$

with boundary conditions

$$\begin{aligned} x_1(0) &= x_{1_0} & x_1(T) &= x_{1_T} \\ y_1(0) &= y_{1_0} & y_2(T) &= 0 \\ z_1(0) &= z_{1_0} & z_2(T) &= 0 \end{aligned} \quad (5.4)$$

and a transversality condition

$$H(x, y, z)|_{t=T} = 0 \quad (5.5)$$

where x_1 , y_1 and z_1 are n , m_1 and m_2 dimensional state variables, x_2 , y_2 and z_2 are n , m_1 and m_2 dimensional costate variables corresponding to x_1 , y_1 and z_1 , $H(x, y, z)$ is a scalar function and ϵ_i for $i = 1, 2$ is a small positive parameter.

The above TPBVP (5.1) – (5.5) comes from application of Pontryagin's minimum principle to a final time unspecified optimal control problem for a nonlinear system with two small parameters. O'Malley developed an asymptotic expansion method in an initial value problem for a nonlinear system with two small parameters [30]. Also, Khalil investigated stability of the full system from stability of the reduced system in a large scaled nonlinear system with multiple small parameters [31] [32]. In this section, we will extend the asymptotic expansion method developed in Chapter III to a nonlinear singularly perturbed TPBVP with multiple parameters.

In (5.1) – (5.5), by setting ϵ_1 and ϵ_2 to zero, we obtain the reduced problem

$$\dot{x}_r = f(x_r, y_r, z_r) \quad (5.6)$$

$$0 = g(x_r, y_r, z_r) \quad (5.7)$$

$$0 = h(x_r, y_r, z_r) \quad (5.8)$$

with

$$x_{1r}(0) = x_{1_0} \quad x_{1r}(T_r) = x_{1T}$$

$$H(x_r, y_r, z_r)|_{t=T_r} = 0.$$

The objective in this section is to seek an asymptotic series solution based on the reduced solution. In order to further proceed, we need some assumptions similar to the earlier assumptions mentioned in Chapter III.

Assumptions.

B.1. f , g , h and H are infinitely differentiable with respect to all their arguments.

B.2. In equation (5.8), $z_r = \phi(x_r, y_r)$ is a continuously differentiable solution

satisfying

$$0 = h(x_r, y_r, \phi(x_r, y_r))$$

Let $y_r = \psi(x_r)$ be a continuously differentiable solution such that

$$0 = g(x_r, \psi(x_r), \phi(x_r, \psi(x_r)))$$

The reduced problem (5.6) – (5.8) has a unique and infinitely differentiable solution on the interval $0 \leq t \leq T_r$.

B.3. The reduced solution satisfies the sufficient conditions for a local minimum.

B.4. There exists a $2m_1 \times 2m_1$ continuously differentiable matrix $M(t)$ for all $0 \leq t \leq T_r$ such that

$$M^{-1}(t)g_{y_r}(x_r, y_r, \phi(x_r, y_r))M(t) = \begin{bmatrix} B(t) & 0 \\ 0 & -B^T(t) \end{bmatrix}$$

Moreover, $M_{11}(0)$ and $M_{22}(T_r)$ are nonsingular and all the real parts of the eigenvalues of $B(t)$ are negative for all $0 \leq t \leq T_r$.

B.5. There exists a $2m_2 \times 2m_2$ continuously differentiable matrix $N(t)$ for all $0 \leq t \leq T_r$ such that

$$N^{-1}(t)h_{z_r}(x_r, y_r, z_r)N(t) = \begin{bmatrix} C(t) & 0 \\ 0 & -C^T(t) \end{bmatrix}$$

Moreover, $N_{11}(0)$ and $N_{22}(T_r)$ are nonsingular and all the real parts of the eigenvalues of $C(t)$ are negative for all $0 \leq t \leq T_r$.

To this end, we will obtain an asymptotic double expansion in ϵ_1 and ϵ_2 in five different time scales. We will define four stretched time scales by

$$\tau_1 = \frac{t}{\epsilon_1} \quad \tau_2 = \frac{t}{\epsilon_1 \epsilon_2} \quad \sigma_1 = \frac{T-t}{\epsilon_1} \quad \sigma_2 = \frac{T-t}{\epsilon_1 \epsilon_2}$$

We want to seek an asymptotic series solution of the form

$$\begin{aligned} x(t, \epsilon_1, \epsilon_2) &= x^O(t, \epsilon_1, \epsilon_2) + \epsilon_1 \left(x^{L_1}(\tau_1, \epsilon_1, \epsilon_2) + x^{R_1}(\sigma_1, \epsilon_1, \epsilon_2) \right) \\ &\quad + \epsilon_1 \epsilon_2 \left(x^{L_2}(\tau_2, \epsilon_1, \epsilon_2) + x^{R_2}(\sigma_2, \epsilon_1, \epsilon_2) \right) \end{aligned} \quad (5.9)$$

$$\begin{aligned} y(t, \epsilon_1, \epsilon_2) &= y^O(t, \epsilon_1, \epsilon_2) + \left(y^{L_1}(\tau_1, \epsilon_1, \epsilon_2) + y^{R_1}(\sigma_1, \epsilon_1, \epsilon_2) \right) \\ &\quad + \epsilon_2 \left(y^{L_2}(\tau_2, \epsilon_1, \epsilon_2) + y^{R_2}(\sigma_2, \epsilon_1, \epsilon_2) \right) \end{aligned} \quad (5.10)$$

$$\begin{aligned} z(t, \epsilon_1, \epsilon_2) &= z^O(t, \epsilon_1, \epsilon_2) + \left(z^{L_1}(\tau_1, \epsilon_1, \epsilon_2) + z^{R_1}(\sigma_1, \epsilon_1, \epsilon_2) \right) \\ &\quad + \left(z^{L_2}(\tau_2, \epsilon_1, \epsilon_2) + z^{R_2}(\sigma_2, \epsilon_1, \epsilon_2) \right) \end{aligned} \quad (5.11)$$

Here, all the terms on the right hand side of (5.9) – (5.11) have an asymptotic double power series expansions in ϵ_1 and ϵ_2 as both ϵ_1 and ϵ_2 go to zero. For example,

$$\begin{aligned} x^O(t, \epsilon_1, \epsilon_2) &= \sum_{j,k \geq 0} x_{jk}^O(t) \epsilon_1^j \epsilon_2^k \\ T &= \sum_{j,k \geq 0} T_{jk} \epsilon_1^j \epsilon_2^k. \end{aligned}$$

In this case, x_{00}^O , y_{00}^O , z_{00}^O and T_{00} will satisfy the reduced problem (5.6) – (5.8).

In the outer region, by substituting $x^O(t, \epsilon_1, \epsilon_2)$, $y^O(t, \epsilon_1, \epsilon_2)$ and $z^O(t, \epsilon_1, \epsilon_2)$ into (5.1) – (5.5) and equating coefficients of equal powers of $\epsilon_1^j \epsilon_2^k$ for $j + k > 0$, we have

$$\begin{aligned} \dot{x}_{jk}^O &= f_x(x_{00}^O, y_{00}^O, z_{00}^O) x_{jk}^O + f_y(x_{00}^O, y_{00}^O, z_{00}^O) y_{jk}^O \\ &\quad + f_z(x_{00}^O, y_{00}^O, z_{00}^O) z_{jk}^O + R_{1,jk}^O(t) \end{aligned} \quad (5.12)$$

$$\begin{aligned} 0 &= g_x(x_{00}^O, y_{00}^O, z_{00}^O) x_{jk}^O + g_y(x_{00}^O, y_{00}^O, z_{00}^O) y_{jk}^O \\ &\quad + g_z(x_{00}^O, y_{00}^O, z_{00}^O) z_{jk}^O + R_{2,jk}^O(t) \end{aligned} \quad (5.13)$$

$$0 = h_x(x_{00}^O, y_{00}^O, z_{00}^O) x_{jk}^O + h_y(x_{00}^O, y_{00}^O, z_{00}^O) y_{jk}^O$$

$$+h_z(x_{00}^O, y_{00}^O, z_{00}^O)z_{jk}^O + R_{3,jk}^O(t) \quad (5.14)$$

$$0 = H_x(x_{00}^O, y_{00}^O, z_{00}^O)x_{jk}^O(T_{00}) + R_{4,jk}^O T_{jk} + R_{5,jk}^O \quad (5.15)$$

where $R_{i,jk}^O$ for $1 \leq i \leq 5$ are known successively in terms of x_{mn}^O , y_{mn}^O , z_{mn}^O and T_{mn} with $m+n < j+k$.

With the aid of Assumptions B.4 and B.5, z_{jk}^O can be obtained in terms of x_{jk}^O and y_{jk}^O from (5.14) and y_{jk}^O can be obtained in terms of x_{jk}^O from (5.13). In order for an asymptotic solution to satisfy the boundary condition (5.4), we require that for $j+k > 0$,

$$0 = x_{1,jk}^O(0) + x_{1,j-1,k}^{L_1}(0) + x_{1,j-1,k-1}^{L_2}(0) \quad (5.16)$$

$$0 = x_{1,jk}^O(T_{00}) + x_{1,j-1,k}^{R_1}(0) + x_{1,j-1,k-1}^{R_2}(0) + R_{6,jk}^O T_{jk} + R_{7,jk}^O \quad (5.17)$$

where $R_{6,jk}^O$ and $R_{7,jk}^O$ are successively known values and quantities with a negative index are defined to be zero.

In the first left boundary layer (τ_1 scale), we will substitute (5.10) – (5.11) into (5.1) – (5.3) assuming that the terms in τ_2 , σ_1 and σ_2 scale are negligible. Then the first boundary layer terms must satisfy

For $j = k = 0$,

$$\begin{aligned} \frac{dx_{00}^{L_1}}{d\tau_1} &= f(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1)) \\ &\quad - f(x_{00}^O(0), y_{00}^O(0), z_{00}^O(0)) \end{aligned} \quad (5.18)$$

$$\begin{aligned} \frac{dy_{00}^{L_1}}{d\tau_1} &= g(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1)) \\ &\quad - g(x_{00}^O(0), y_{00}^O(0), z_{00}^O(0)) \end{aligned} \quad (5.19)$$

$$\begin{aligned}
0 &= h(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1)) \\
&\quad - h(x_{00}^O(0), y_{00}^O(0), z_{00}^O(0))
\end{aligned} \tag{5.20}$$

For $j + k > 0$,

$$\begin{aligned}
\frac{dx_{jk}^{L_1}}{d\tau_1} &= f_y(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))y_{jk}^{L_1}(\tau_1) \\
&\quad + f_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))z_{jk}^{L_1}(\tau_1) \\
&\quad + R_{1,jk}^{L_1}(\tau_1)
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
\frac{dy_{jk}^{L_1}}{d\tau_1} &= g_y(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))y_{jk}^{L_1}(\tau_1) \\
&\quad + f_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))z_{jk}^{L_1}(\tau_1) \\
&\quad + R_{2,jk}^{L_1}(\tau_1)
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
0 &= h_y(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))y_{jk}^{L_1}(\tau_1) \\
&\quad + h_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(\tau_1), z_{00}^O(0) + z_{00}^{L_1}(\tau_1))z_{jk}^{L_1}(\tau_1) \\
&\quad + R_{3,jk}^{L_1}(\tau_1)
\end{aligned} \tag{5.23}$$

with the initial condition

$$\begin{aligned}
y_{100}^{L_1}(0) &= y_{10} - y_{100}^O(0) \\
y_{1,k}^{L_1}(0) &= -y_{1,k}^O(0) - y_{1,j,k-1}^{L_2}(0)
\end{aligned}$$

where $R_{i,jk}^{L_1}(\tau_1)$ for $1 \leq i \leq 3$ are successively known functions and decay to zero as τ_1 goes to infinity.

From (5.20) or (5.23), $z_{jk}^{L_1}$ can be obtained in terms of $y_{jk}^{L_1}$ algebraically. Hence (5.19) and (5.22) is a differential equation containing only $y_{jk}^{L_1}$. The initial condition of the fast costate $y_{2,k}^{L_1}(0)$ must be so chosen that $y_{jk}^{L_1}$ decays exponentially to zero as τ_1 goes to infinity. Then, since the right sides of (5.18) and (5.21) are known, $x_{jk}^{L_1}$ can be obtained by integration. The exponential stability in the first left boundary layer is guaranteed if Assumption B.4 holds and $|y_{1,k}^{L_1}(0)|$ is sufficiently small.

Similarly, in the second left boundary layer (τ_2 scale), the second left boundary layer terms must satisfy

For $j = k = 0$,

$$\begin{aligned} \frac{dx_{00}^{L_2}}{d\tau_2} &= f(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2)) \\ &\quad - f(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0)) \end{aligned} \quad (5.24)$$

$$\begin{aligned} \frac{dy_{00}^{L_2}}{d\tau_2} &= g(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2)) \\ &\quad - g(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0)) \end{aligned} \quad (5.25)$$

$$\begin{aligned} \frac{dz_{00}^{L_2}}{d\tau_2} &= h(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2)) \\ &\quad - h(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O(0) + z_{00}^{L_1}(0)). \end{aligned} \quad (5.26)$$

For $j + k > 0$,

$$\begin{aligned} \frac{dx_{jk}^{L_2}}{d\tau_2} &= f_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2)) z_{jk}^{L_2}(\tau_2) \\ &\quad + R_{1,jk}^{L_2}(\tau_2) \end{aligned} \quad (5.27)$$

$$\begin{aligned} \frac{dy_{jk}^{L_2}}{d\tau_2} &= g_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2)) z_{jk}^{L_2}(\tau_2) \\ &\quad + R_{2,jk}^{L_2}(\tau_2) \end{aligned} \quad (5.28)$$

$$\begin{aligned}
\frac{dz_{jk}^{L_2}}{d\tau_2} &= h_z(x_{00}^O(0), y_{00}^O(0) + y_{00}^{L_1}(0), z_{00}^O + z_{00}^{L_1}(0) + z_{00}^{L_2}(\tau_2))z_{jk}^{L_2}(\tau_2) \\
&\quad + R_{3,jk}^{L_2}(\tau_2)
\end{aligned} \tag{5.29}$$

with the initial condition

$$\begin{aligned}
z_{100}^{L_2}(0) &= z_{10} - z_{100}^O(0) - z_{100}^{L_1}(0) \\
z_{1,jk}^{L_2}(0) &= -z_{1,jk}^O(0) - z_{1,jk}^{L_1}(0)
\end{aligned}$$

where $R_{i,jk}^{L_2}(\tau_2)$ for $1 \leq i \leq 3$ are successively known functions and decay to zero as τ_2 goes to infinity. By Assumption B.5, we can choose $z_{2,jk}^{L_2}(0)$ such that $z_{jk}^{L_2}(\tau_2)$ decays exponentially to zero as τ_2 goes to infinity if $|z_{2,jk}^{L_2}(0)|$ is sufficiently small.

By the same manner, the first and the second right boundary layer expansion can be obtained. Moreover, extension of this asymptotic expansion method to a nonlinear TPBVP

$$\dot{x} = f(x, y_1, \dots, y_n)$$

$$\epsilon_1 \epsilon_2 \cdots \epsilon_k \dot{y}_k = g(x, y_1, \dots, y_n) \quad 1 \leq k \leq n$$

with n small positive parameters ϵ_k for $1 \leq k \leq n$ will be possible by a straightforward manner.

CHAPTER VI

APPLICATION OF SINGULAR PERTURBATION TO BTT MISSILE GUIDANCE

6.1 Problem Statement

Preliminaries.

The proportional navigation (PN) guidance law has been widely used for a missile guidance during the past several decades because of its effectiveness and simplicity of implementation. In the PN guidance law, the acceleration command is generated by

$$a_c = K_{nav} v_c \dot{\sigma}$$

where a_c is the acceleration command, v_c is a closing velocity, σ is a line of sight (LOS) angle. The navigation constant K_{nav} is determined experimentally. It was shown that the PN guidance law was optimal under certain conditions [25]. The PN guidance law, however, is not generally effective against maneuvering targets [7].

In order to obtain better performance, many authors have attempted to derive a guidance law by using modern control theory. Actually, the guidance problem for an air-to-air missile may be formulated as an optimal control problem. Since the missile motion equation is nonlinear, the resulting nonlinear TPBVP is difficult to solve by on-line computation.

In order to circumvent this computational burden, Fiske developed an optimal linear (OL) guidance law by using a simplified missile motion equation modelled as

double integrator in the inertial axis [33]. If a missile is modelled as a point mass in the inertial axis, the missile motion equation can be casted into the following form.

$$\dot{x} = v_x$$

$$\dot{v}_x = a_x$$

where x , v_x and a_x are the relative distance, relative velocity between the missile and a target and an acceleration command in the inertial X-axis.

The OL guidance law was obtained by minimizing the following quadratic performance index.

$$J = \int_0^{T_{go}} \frac{1}{2} a_x^2 dt + \frac{1}{2} Q x(T_{go})^2$$

where T_{go} is computed by an appropriate time-to-go estimation algorithm. The inertial X-axis acceleration command can be easily obtained in a closed form. By the same method, the inertial Y-axis and Z-axis acceleration command can be computed. From these inertial acceleration commands, an acceleration command in the body axis can be obtained by the inertial to the body axis coordinate transformation. The desired roll angle command is computed such that the body Y-axis acceleration command is zero. Since the body X-axis acceleration is nearly constant in our concerned BTT missile, the computed body X-axis acceleration command should be neglected in Fiske's OL guidance law. The performance of the optimal linear guidance law is satisfactory. However, when the initial range is short (within 4000 ft), the performance of the OL guidance law is not satisfactory. This may be due to the neglected missile body X-axis acceleration.

Recently, an attempt has been made to derive a nearly optimal guidance law from a nonlinear missile motion equation by the singular perturbation method.

Sridar derived a nearly optimal guidance law which is approximated up to the zeroth order [7]. But, Sridar observed that the solution significantly deviates from the optimal as time-to-go decreases.

In order to obtain a better approximation, Visser suggested a first order correction method by using the matched asymptotic expansion method [9]. But, it was observed that the nearly optimal guidance law derived based on the singular perturbation method also failed to give a good approximation during the terminal phase in which the time scale decomposition is not valid.

Problem Statement.

We define the reference axis as the wind axis at each sampling time. Then, the missile motion equations in the reference axis will be

$$\frac{dx_w}{dt} = V \cos \gamma \cos \psi - V_{txw} \quad (6.1)$$

$$\frac{dy_w}{dt} = V \cos \gamma \sin \psi - V_{tyw} \quad (6.2)$$

$$\frac{dz_w}{dt} = V \sin \gamma - V_{tzw} \quad (6.3)$$

$$\frac{d\gamma}{dt} = \cos \phi \frac{L_w}{V} \quad (6.4)$$

$$\frac{d\psi}{dt} = \sin \phi \frac{L_w}{V \cos \gamma} \quad (6.5)$$

$$\frac{d\phi}{dt} = p_w \quad (6.6)$$

with the initial condition

$$x_w(0) = x_{0w} \quad y_w(0) = y_{0w} \quad z_w(0) = z_{0w}$$

$$\gamma(0) = 0 \quad \psi(0) = 0 \quad \phi(0) = 0$$

where

x_w, y_w, z_w : the relative distance

$V_{txw}, V_{tyw}, V_{tzw}$: the target velocity (assumed to be constant)

V : the missile average velocity

γ, ψ, ϕ : the flight path, heading and roll angle

L_w : the vertical lift

p_w : the roll rate

$|L_w| \leq L_{wmax}, |p_w| \leq p_{wmax}$.

In this chapter, our aim will be

1. to derive a nearly optimal guidance law with the first order correction terms using the method described in Chapter V.
2. to increase the region in which the time scaling decomposition is valid by adjusting control weighting.

6.2 Singularly Perturbed TPBVP

This section contains a procedure to transform (6.1) – (6.6) into a standard form of a singularly perturbed system. In order to formulate the problem in the form for singular perturbations, we normalize the variables and time scale as follows.

Let

$$R = (x_{0w}^2 + y_{0w}^2 + z_{0w}^2)^{\frac{1}{2}}$$

$$x = x_w/R \quad y = y_w/R \quad z = z_w/R$$

$$V_{tx} = V_{txw}/R \quad V_{ty} = V_{tyw}/R \quad V_{tz} = V_{tzw}/R$$

$$\tilde{L} = L_w/L_{wmax} \quad \tilde{p} = p_w/p_{wmax} \quad t' = Vt/R.$$

Using the above definitions, equations (6.1) – (6.6) become

$$\dot{x} = \cos \gamma \cos \psi - V_{tx} \quad (6.7)$$

$$\dot{y} = \cos \gamma \sin \psi - V_{ty} \quad (6.8)$$

$$\dot{z} = \sin \gamma - V_{tz} \quad (6.9)$$

$$\mu_1 \dot{\gamma} = \cos \phi \bar{L} \quad (6.10)$$

$$\mu_1 \dot{\psi} = \sin \phi \frac{\bar{L}}{\cos \gamma} \quad (6.11)$$

$$\mu_1 \mu_2 \dot{\phi} = \bar{p} \quad (6.12)$$

with the initial condition

$$x(0) = x_{0w}/R = x_0 \quad y(0) = y_{0w}/R = y_0 \quad z(0) = z_{0w}/R = z_0$$

$$\gamma(0) = 0 \quad \psi(0) = 0 \quad \phi(0) = 0$$

where

$$\dot{x} = \frac{dx}{dt'}$$

$$\mu_1 = \frac{V^2}{RL_{wmax}}$$

$$\mu_2 = \frac{L_{wmax}}{p_{max}V}.$$

The controls $\bar{L}$ and $\bar{p}$ will be chosen to minimize a quadratic performance index with a cheap control cost.

$$J = \int_0^T (1 + \frac{1}{2} k_1^2 \bar{L}^2 + \frac{1}{2} k_1^2 k_2^2 \bar{p}^2) dt'$$

subject to the final state and control constraints for an unspecified T .

$$i) \quad x(T) = y(T) = z(T) = 0$$

$$ii) \quad |\bar{L}| \leq 1 \quad |\bar{p}| \leq 1$$

where $0 < k_1, k_2 \ll 1$.

The Hamiltonian for this problem is

$$\begin{aligned} H = & \bar{\lambda}_x(\cos \gamma \cos \psi - V_{tx}) + \bar{\lambda}_y(\cos \gamma \sin \psi - V_{ty}) + \bar{\lambda}_z(\sin \gamma - V_{tz}) \\ & + \bar{\lambda}_\gamma \cos \phi \bar{L} + \bar{\lambda}_\psi \sin \phi \frac{\bar{L}}{\cos \gamma} + \bar{\lambda}_\phi \bar{p} \\ & + 1 + \frac{1}{2} k_1^2 \bar{L}^2 + \frac{1}{2} k_1^2 k_2^2 \bar{p}^2 \end{aligned}$$

where for example, $\bar{\lambda}_x$ denotes the costate corresponding to x .

Now, we define new controls and new costates as follows.

$$\begin{aligned} L &= k_1 \bar{L} \quad p = k_1 k_2 \bar{p} \quad \lambda_x = \bar{\lambda}_x \\ \lambda_y &= \bar{\lambda}_y \quad \lambda_z = \bar{\lambda}_z \quad \lambda_\gamma = \frac{\bar{\lambda}_\gamma}{k_1} \\ \lambda_\psi &= \frac{\bar{\lambda}_\psi}{k_1} \quad \lambda_\phi = \frac{\bar{\lambda}_\phi}{k_1 k_2}. \end{aligned}$$

Then, from the Euler-Lagrange equation, the necessary conditions for optimality will be

$$\begin{aligned} H = & \lambda_x(\cos \gamma \cos \psi - V_{tx}) + \lambda_y(\cos \gamma \sin \psi - V_{ty}) + \lambda_z(\sin \gamma - V_{tz}) \\ & + \lambda_\gamma \cos \phi L + \lambda_\psi \sin \phi \frac{L}{\cos \gamma} + \lambda_\phi p + \frac{1}{2} L^2 + \frac{1}{2} p^2 + 1 \\ \equiv & 0 \end{aligned} \tag{6.13}$$

$$\dot{x} = \cos \gamma \cos \psi - V_{tx} \tag{6.14}$$

$$\dot{y} = \cos \gamma \sin \psi - V_{ty} \tag{6.15}$$

$$\dot{z} = \sin \gamma - V_{tz} \tag{6.16}$$

$$\epsilon_1 \dot{\gamma} = \cos \phi L \quad (6.17)$$

$$\epsilon_1 \dot{\psi} = \frac{\sin \phi L}{\cos \gamma} \quad (6.18)$$

$$\epsilon_1 \epsilon_2 \dot{\phi} = p \quad (6.19)$$

$$\dot{\lambda}_x = \dot{\lambda}_y = \dot{\lambda}_z = 0 \quad (6.20)$$

$$\epsilon_1 \dot{\lambda}_\gamma = -\frac{\partial H}{\partial \gamma} \quad (6.21)$$

$$\epsilon_1 \dot{\lambda}_\psi = -\frac{\partial H}{\partial \psi} \quad (6.22)$$

$$\epsilon_1 \epsilon_2 \dot{\lambda}_\phi = -\frac{\partial H}{\partial \phi} \quad (6.23)$$

$$L = \arg \min_{|L| \leq k_1} H \quad (6.24)$$

$$p = \arg \min_{|p| \leq k_1 k_2} H \quad (6.25)$$

where

$$\epsilon_1 = \mu_1 k_1 \quad \epsilon_2 = \mu_2 k_2$$

$$x(0) = x_0 \quad y(0) = y_0 \quad z(0) = z_0$$

$$\gamma(0) = 0 \quad \psi(0) = 0 \quad \phi(0) = 0.$$

Remark 6.1. Actually, the accuracy of an asymptotic solution depends on the size of the singular perturbation parameter ϵ . But in an actual system to be controlled, it is difficult to identify the exact value of the singular perturbation parameter. Usually, a designer can estimate a singular perturbation parameter roughly based on his insight into a dynamic system or rough estimates of the time derivatives of the states as in deriving (6.7) – (6.12).

Remark 6.2. In equations (6.7) – (6.12), the singular perturbation parameter μ_1 and μ_2 may not be valid during the whole interval. For example, μ_1 becomes bigger as the time-to-go goes to zero (the relative distance is shorter). In order to compensate rough estimates of μ_1 and μ_2 , we introduced a performance index with a cheap control cost. In the TPBVP (6.13) – (6.25), the singular perturbation parameter ϵ_1 and ϵ_2 are functions of μ_1, μ_2, k_1, k_2 . By choosing small k_1 and k_2 , the singular perturbation parameter ϵ_1 and ϵ_2 can be arbitrarily small.

Remark 6.3. Introducing a small control weighting in a performance index can be found in earlier papers. O'Malley and Jameson utilized this scheme for a singular linear quadratic optimal control problem [34] [35]. In order to seek an optimal solution in a singular optimal control problem, they regularized the problem by introducing ϵ in a control cost and obtained an asymptotic solution by the singular perturbation method. Ardema also investigated the behaviour of the solution of a nonlinear singular control problem on the singular arc by regularizing the initial problem [36].

6.3 Derivation of the Guidance Law

In this section, we will derive an asymptotic solution of the nonlinear TPBVP (6.13) – (6.25) in a closed form. In obtaining the asymptotic solution, the most difficult part of the solution procedure is to solve the zeroth order boundary layer equations whose solution must be asymptotic to the zeroth order outer solution. Once the zeroth order solution is obtained, the first order correction terms can be made based on a procedure described in Chapter III.

A linearization approach [37][38] for solving the nonlinear zeroth order boundary layer equations was suggested. But, in this case, when initial conditions of fast states

are far away from the zeroth order outer solution, there is no guarantee such that the initial conditions are in the region of attraction. When i) nonlinear boundary layer equation is scalar, ii) the final time is unspecified and iii) the Hamiltonian is not an explicit function of time, a closed form solution can be easily obtained from the fact that the Hamiltonian is identically zero. In this section, we will use a coordinate transformation such that the 2 dimensional boundary layer equations in the first left boundary layer can be treated as a scalar boundary layer equation.

Outer Solution.

By setting ϵ_1 and ϵ_2 to zero in the outer expansion, (6.14) – (6.23) become

$$\dot{x}_{00}^O = \cos \gamma_{00}^O \cos \psi_{00}^O - V_{tx} \quad x_{00}^O(0) = x_0 \quad x_{00}^O(T_{00}) = 0 \quad (6.26)$$

$$\dot{y}_{00}^O = \cos \gamma_{00}^O \sin \psi_{00}^O - V_{ty} \quad y_{00}^O(0) = y_0 \quad y_{00}^O(T_{00}) = 0 \quad (6.27)$$

$$\dot{z}_{00}^O = \sin \gamma_{00}^O - V_{tz} \quad z_{00}^O(0) = z_0 \quad z_{00}^O(T_{00}) = 0 \quad (6.28)$$

$$0 = \cos \phi_{00}^O L_{00}^O \quad (6.29)$$

$$0 = \sin \phi_{00}^O \frac{L_{00}^O}{\cos \gamma_{00}^O} \quad (6.30)$$

$$0 = p_{00}^O \quad (6.31)$$

$$\dot{\lambda}_{x00}^O = \dot{\lambda}_{y00}^O = \dot{\lambda}_{z00}^O = 0 \quad (6.32)$$

$$\begin{aligned} 0 = & \lambda_{x00}^O \sin \gamma_{00}^O \cos \psi_{00}^O + \lambda_{y00}^O \sin \gamma_{00}^O \sin \psi_{00}^O - \lambda_{x00}^O \cos \gamma_{00}^O \\ & - \lambda_{y00}^O \sin \phi_{00}^O \frac{\sin \gamma_{00}^O}{\cos^2 \gamma_{00}^O} L_{00}^O \end{aligned} \quad (6.33)$$

$$0 = \lambda_{x00}^O \cos \gamma_{00}^O \sin \psi_{00}^O - \lambda_{y00}^O \cos \gamma_{00}^O \cos \psi_{00}^O \quad (6.34)$$

$$0 = \lambda_{x00}^O \sin \phi_{00}^O L_{00}^O - \lambda_{y00}^O \cos \phi_{00}^O \frac{L_{00}^O}{\cos \gamma_{00}^O}. \quad (6.35)$$

From (6.34), ψ_{00}^O is constant. From (6.29) and (6.30), $L_{00}^O \equiv 0$. Then from (6.33), γ_{00}^O is also constant.

Hence, from (6.26) – (6.28), we can compute T_{00} , γ_{00}^O and ψ_{00}^O , i.e.,

$$T_{00} = \frac{-C + \sqrt{C^2 + AB}}{A} \quad (6.36)$$

$$\gamma_{00}^O(t) \equiv \tan^{-1} \frac{V_{tx}T_{00} - z_0}{\sqrt{(V_{tx}T_{00} - x_0)^2 + (V_{ty}T_{00} - y_0)^2}} \quad (6.37)$$

$$\psi_{00}^O \equiv \tan^{-1} \frac{V_{ty}T_{00} - y_0}{V_{tx}T_{00} - x_0} \quad (6.38)$$

where

$$A = 1 - (V_{tx}^2 + V_{ty}^2 + V_{tz}^2)$$

$$B = x_0^2 + y_0^2 + z_0^2$$

$$C = V_{tx}x_0 + V_{ty}y_0 + V_{tz}z_0.$$

We can obtain further simplification by redefining the roll origin, that is we perform a rotation about the X-axis by an angle $\Delta\phi$. Then,

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \mathcal{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Delta\phi & \sin \Delta\phi \\ 0 & -\sin \Delta\phi & \cos \Delta\phi \end{bmatrix}.$$

In the new coordinate system, $\hat{\gamma}$, $\hat{\psi}$ and $\hat{\phi}$ are defined as before with respect to the new axis. Since the functional form of the equations is unchanged, we have

$$\hat{\psi}_{00}^O \equiv \tan^{-1} \frac{\hat{V}_{ty}T_{00} - \hat{y}_0}{\hat{V}_{tx}T_{00} - \hat{x}_0}.$$

By a choice of $\Delta\phi$, we can force $\hat{\psi}_{00}^O \equiv 0$ as follows.

$$\begin{aligned}\hat{V}_{ty}T_{00} - \hat{y}_0 &= (\cos \Delta\phi V_{ty} + \sin \Delta\phi V_{tz})T_{00} - (\cos \Delta\phi y_0 + \sin \Delta\phi z_0) \\ &\equiv 0\end{aligned}$$

if $\Delta\phi$ is chosen to satisfy

$$\Delta\phi = -\tan^{-1} \frac{y_0 - V_{ty}T_0}{z_0 - V_{tz}T_0}.$$

In the new axis,

$$\begin{aligned}\hat{\gamma}_{00}^O(t) &\equiv \tan^{-1} \frac{\hat{V}_{tz}T_{00} - \hat{z}_0}{\hat{V}_{tx}T_{00} - \hat{x}_0} \\ \hat{\gamma}(0) = \hat{\psi}(0) &= 0 \quad \hat{\phi}(0) = -\Delta\phi.\end{aligned}\tag{6.39}$$

In the following, we choose the new variables by x, y, z, γ, ψ and ϕ in place of $\hat{x}, \hat{y}, \hat{z}, \hat{\gamma}, \hat{\psi}$ and $\hat{\phi}$ for simplicity.

Also, the $H \equiv 0$ equation becomes

$$\begin{aligned}H_{00}^O &= 1 + \lambda_{x00}^O(\cos \gamma_{00}^O \cos \psi_{00}^O - V_{tx}) + \lambda_{y00}^O(\cos \gamma_{00}^O \sin \psi_{00}^O - V_{ty}) \\ &\quad + \lambda_{z00}^O(\sin \gamma_{00}^O - V_{tz}) \equiv 0\end{aligned}\tag{6.40}$$

From (6.33), (6.34) and (6.40),

$$\lambda_{x00}^O = \frac{-\cos \gamma_{00}^O}{1 - (V_{tx} \cos \gamma_{00}^O + V_{tz} \sin \gamma_{00}^O)}\tag{6.41}$$

$$\lambda_{y00}^O \equiv 0\tag{6.42}$$

$$\lambda_{z00}^O = \lambda_{x00}^O \tan \gamma_{00}^O.\tag{6.43}$$

First Left Boundary Layer Solution.

Since γ_{00}^O in (6.39) does not satisfy the initial condition $\gamma(0)$ in general, boundary layer analysis is required to eliminate this discrepancy.

Since ψ_{00}^O , ϕ_{00}^O , L_{00}^O and p_{00}^O are identically zero, using the time scale τ_1 gives the following set of the zeroth order necessary conditions in the first boundary layer.

$$\frac{dx_{00}^{L_1}}{d\tau_1} = \cos(\gamma_{00}^O + \gamma_{00}^{L_1}) \cos \psi_{00}^{L_1} - \cos \gamma_{00}^O \quad (6.44)$$

$$\frac{dy_{00}^{L_1}}{d\tau_1} = \cos(\gamma_{00}^O + \gamma_{00}^{L_1}) \sin \psi_{00}^{L_1} \quad (6.45)$$

$$\frac{dz_{00}^{L_1}}{d\tau_1} = \sin(\gamma_{00}^O + \gamma_{00}^{L_1}) - \sin \gamma_{00}^O \quad (6.46)$$

$$\frac{d\lambda_{x00}^{L_1}}{d\tau_1} = \frac{d\lambda_{y00}^{L_1}}{d\tau_1} = \frac{d\lambda_{z00}^{L_1}}{d\tau_1} = 0 \quad (6.47)$$

$$\frac{d\gamma_{00}^{L_1}}{d\tau_1} = \cos \phi_{00}^{L_1} L_{00}^{L_1} \quad (6.48)$$

$$\frac{d\psi_{00}^{L_1}}{d\tau_1} = \sin \phi_{00}^{L_1} \frac{L_{00}^{L_1}}{\cos(\gamma_{00}^O + \gamma_{00}^{L_1})} \quad (6.49)$$

$$0 = p_{00}^{L_1} \quad (6.50)$$

$$\begin{aligned} \frac{d\lambda_{\gamma 00}^{L_1}}{d\tau_1} &= \lambda_{x00}^O \sin(\gamma_{00}^O + \gamma_{00}^{L_1}) \cos \psi_{00}^{L_1} - \lambda_{z00}^O \cos(\gamma_{00}^O + \gamma_{00}^{L_1}) \\ &\quad - \lambda_{\psi 00}^{L_1} \sin \phi_{00}^{L_1} \sin(\gamma_{00}^O + \gamma_{00}^{L_1}) \frac{L_{00}^{L_1}}{\cos^2(\gamma_{00}^O + \gamma_{00}^{L_1})} \end{aligned} \quad (6.51)$$

$$\frac{d\lambda_{\psi 00}^{L_1}}{d\tau_1} = \lambda_{x00}^O \cos(\gamma_{00}^O + \gamma_{00}^{L_1}) \sin \psi_{00}^{L_1} \quad (6.52)$$

$$0 = \lambda_{\gamma 00}^{L_1} \sin \phi_{00}^{L_1} L_{00}^{L_1} - \lambda_{\psi 00}^{L_1} \cos \phi_{00}^{L_1} \frac{L_{00}^{L_1}}{\cos(\gamma_{00}^O + \gamma_{00}^{L_1})} \quad (6.53)$$

$$\gamma_{00}^{L_1}(0) = \gamma(0) - \gamma_{00}^O(0) = -\gamma_{00}^O(0)$$

$$\psi_{00}^{L_1}(0) = \psi(0) - \psi_{00}^O(0) = 0.$$

From (6.47), we have

$$\lambda_{x00}^{L_1} \equiv 0 \quad \lambda_{y00}^{L_1} \equiv 0 \quad \lambda_{z00}^{L_1} \equiv 0 \quad (6.54)$$

Since $\psi_{00}^{L_1}(0) = 0$, no boundary layer transition $\psi_{00}^{L_1}$ can be expected. This fact

motivates the coordinate transformation described earlier in this section. Setting $\phi_{00}^{L_1}(\tau_1) \equiv 0$ in (6.49) gives $\psi_{00}^{L_1}(\tau_1) \equiv 0$. Hence by using (6.54) and

$$\psi_{00}^{L_1}(\tau_1) = \phi_{00}^{L_1}(\tau_1) \equiv 0,$$

the Hamiltonian (6.13) in the first boundary layer will be

$$\begin{aligned} H_{00}^{L_1} &= \lambda_{x00}^O (\cos(\gamma_{00}^{L_1} + \gamma_{00}^O) - V_{tx}) + \lambda_{z00}^O (\sin(\gamma_{00}^{L_1} + \gamma_{00}^O) - V_{tz}) \\ &\quad + \lambda_{\gamma_{00}}^{L_1} L_{00}^{L_1} + 1 + \frac{1}{2} (L_{00}^{L_1})^2 \equiv 0. \end{aligned} \quad (6.55)$$

Since $H_{00}^O = H_{00}^{L_1}$, combining (6.40) and (6.55) gives

$$\begin{aligned} \lambda_{\gamma_{00}}^{L_1} L_{00}^{L_1} + \frac{1}{2} (L_{00}^{L_1})^2 &= \lambda_{x00}^O (\cos \gamma_{00}^O - \cos(\gamma_{00}^{L_1} + \gamma_{00}^O)) \\ &\quad + \lambda_{z00}^O (\sin \gamma_{00}^O - \sin(\gamma_{00}^{L_1} + \gamma_{00}^O)) \\ &= \frac{-2}{1 - (V_{tx} \cos \gamma_{00}^O + V_{tz} \sin \gamma_{00}^O)} \sin^2 \frac{\gamma_{00}^{L_1}}{2}. \end{aligned} \quad (6.56)$$

From (6.24),

$$L_{00}^{L_1} = -k_1 \text{sat}\left(\frac{\lambda_{\gamma_{00}}^{L_1}}{k_1}\right) \quad (6.57)$$

where

$$\text{sat}(f(x)) = \begin{cases} 1 & \text{if } f(x) \geq 1 \\ f(x) & \text{if } -1 \leq f(x) \leq 1 \\ -1 & \text{if } f(x) \leq -1. \end{cases}$$

Hence from (6.56) and (6.57), the unsaturated control $L_{00}^{L_1}$ satisfies

$$L_{00}^{L_1} = K \sin \frac{\gamma_{00}^{L_1}}{2} \quad (6.58)$$

where

$$K = -\frac{2}{\sqrt{1 - (V_{tx} \cos \gamma_{00}^O + V_{tz} \sin \gamma_{00}^O)}}.$$

If the control is saturated (i.e., $|K \sin \frac{\gamma_{00}^{L_1}}{2}| \geq k_1$),

$$L_{00}^{L_1} = -k_1 \operatorname{sign}(\gamma_{00}^{L_1}). \quad (6.59)$$

Since $L_{00}^{L_1}$ is known by (6.58) or (6.59), $\lambda_{\gamma_{00}}^{L_1}$ can be computed from (6.56)

$$\lambda_{\gamma_{00}}^{L_1} = -\frac{\frac{1}{2} K^2 \sin^2 \frac{\gamma_{00}^{L_1}}{2} + \frac{1}{2} (L_{00}^{L_1})^2}{L_{00}^{L_1}} \quad (6.60)$$

Since $\phi_{00}^{L_1} \equiv 0$, from (6.53) we obtain

$$\lambda_{\psi_{00}}^{L_1} \equiv 0.$$

Second Left Boundary Layer Solution.

In the second left boundary layer, application of (5.24) – (5.26) using

$$\psi_{00}^O = \psi_{00}^{L_1} = \lambda_{\gamma_{00}}^O = \lambda_{\psi_{00}}^O = \lambda_{\psi_{00}}^{L_1} = \phi_{00}^O = \phi_{00}^{L_1} = \lambda_{\phi_{00}}^O = \lambda_{\phi_{00}}^{L_1} = L_{00}^O = 0$$

gives

$$\frac{dx_{00}^{L_2}}{d\tau_2} = 0 \quad \frac{dy_{00}^{L_2}}{d\tau_2} = 0 \quad \frac{dz_{00}^{L_2}}{d\tau_2} = 0 \quad (6.61)$$

$$\frac{d\lambda_{x_{00}}^{L_2}}{d\tau_2} = 0 \quad \frac{d\lambda_{y_{00}}^{L_2}}{d\tau_2} = 0 \quad \frac{d\lambda_{z_{00}}^{L_2}}{d\tau_2} = 0 \quad (6.62)$$

$$\frac{d\gamma_{00}^{L_2}}{d\tau_2} = \cos \phi_{00}^{L_2} (L_{00}^{L_1}(0) + L_{00}^{L_2}) - L_{00}^{L_1}(0) \quad (6.63)$$

$$\frac{d\psi_{00}^{L_2}}{d\tau_2} = \sin \phi_{00}^{L_2} \frac{L_{00}^{L_1}(0) + L_{00}^{L_2}}{\cos(\gamma_{00}^O(0) + \gamma_{00}^{L_1}(0))} \quad (6.64)$$

$$\frac{d\lambda_{\gamma_{00}}^{L_2}}{d\tau_2} = 0 \quad (6.65)$$

$$\frac{d\lambda_{\psi_{00}}^{L_2}}{d\tau_2} = 0 \quad (6.66)$$

$$\frac{d\phi_{00}^{L_2}}{d\tau_2} = p_{00}^{L_2} \quad (6.67)$$

$$\frac{d\lambda_{\phi_{00}}^{L_2}}{d\tau_2} = \lambda_{\gamma_{00}}^{L_1}(0) \sin \phi_{00}^{L_2} (L_{00}^{L_1}(0) + L_{00}^{L_2}) \quad (6.68)$$

with the initial condition

$$\phi_{00}^{L_2}(0) = -\Delta\phi.$$

From (6.65) and (6.66),

$$\lambda_{\gamma 00}^{L_2}(\tau_2) \equiv \lambda_{\gamma 00}^{L_1}(0) \quad \lambda_{\psi 00}^{L_2}(\tau_2) \equiv \lambda_{\psi 00}^{L_1}(0) = 0.$$

Hence, from the Hamiltonian in the second left boundary layer

$$\begin{aligned} L_{00}^{L_2} &= \arg \min_{L_{00}^{L_2}} H_{00}^{L_2} \\ &= -k_1 \text{sat} \left(\frac{1}{k_1} \lambda_{\gamma 00}^{L_1} \cos \phi_{00}^{L_2} \right) - L_{00}^{L_1} \end{aligned} \quad (6.69)$$

$$\begin{aligned} p_{00}^{L_2} &= \arg \min_{p_{00}^{L_2}} H_{00}^{L_2} \\ &= -k_1 k_2 \text{sat} \left(\frac{\lambda_{\phi 00}^{L_2}}{k_1 k_2} \right). \end{aligned} \quad (6.70)$$

Since the Hamiltonian is constant, from $H_{00}^{L_2} = H_{00}^{L_1}$ we obtain

$$\begin{aligned} \frac{1}{2} (p_{00}^{L_2})^2 + \lambda_{\psi 00}^{L_2} p_{00}^{L_2} &= -\lambda_{\gamma 00}^{L_1} \cos \phi_{00}^{L_2} (L_{00}^{L_1} + L_{00}^{L_2}) + \lambda_{\gamma 00}^{L_1} L_{00}^{L_1} \\ &\quad - \frac{1}{2} ((L_{00}^{L_1} + L_{00}^{L_2})^2 - (L_{00}^{L_1})^2) \end{aligned} \quad (6.71)$$

Using (6.69) and (6.70), from (6.71)

$$p_{00}^{L_2} = -k_1 k_2 \text{sat} \left(\frac{\sqrt{2P}}{k_1 k_2} \right) \text{sign}(\phi_{00}^{L_2}) \quad (6.72)$$

where

$$P = \lambda_{\gamma 00}^{L_1} \cos \phi_{00}^{L_2} (L_{00}^{L_1} + L_{00}^{L_2}) - \lambda_{\gamma 00}^{L_1} L_{00}^{L_1} + \frac{1}{2} ((L_{00}^{L_1} + L_{00}^{L_2})^2 - (L_{00}^{L_1})^2).$$

We remark that if the zeroth order control is implemented by a feedback scheme, all the zeroth order state variables appeared in (6.69) and (6.72) can be replaced by the exact values since all the state variables are measurable.

First Order Correction.

Up to this point, the zeroth order solutions in the outer region and the first left boundary layer region are derived. These solutions are derived under the assumption that the slow variables are fixed while the fast variables reach to their steady state values. But this assumption may not hold during the terminal phase. Since the singular perturbation parameter ϵ_2 is a function of the maximum allowable limits of control variables and the missile average velocity, ϵ_2 is nearly independent of a missile-target geometry. Hence, we will derive the first order correction terms in ϵ_1 in order to increase the region in which the singularly perturbed solution is valid. As can be seen in (6.56), (6.57) and (6.69), the control variables are functions of only the slow costate variables if all the state variables are measurable. Hence, it suffices to derive the first order correction terms in the slow costate variables.

To this end, for simplicity, we will define $\hat{x}^a = x_{00}^a + \epsilon_1 x_{10}^a$ where the superscript a is 0 or L_1 . In the outer region, (6.14) – (6.23) can be expressed up to the first order in ϵ_1 in the following form if ϵ_1 is sufficiently small.

$$\dot{\hat{x}}^O = \cos \hat{\gamma}^O \cos \hat{\psi}^O - V_{tx} \quad (6.73)$$

$$\dot{\hat{y}}^O = \cos \hat{\gamma}^O \sin \hat{\psi}^O - V_{ty} \quad (6.74)$$

$$\dot{\hat{z}}^O = \sin \hat{\gamma}^O - V_{tz} \quad (6.75)$$

$$0 = \cos \hat{\phi}^O \hat{L}^O \quad (6.76)$$

$$0 = \sin \hat{\phi}^O \frac{\hat{L}^O}{\cos \hat{\gamma}^O} \quad (6.77)$$

$$0 = \hat{p}^O \quad (6.78)$$

$$\dot{\hat{\lambda}}_x^O = \dot{\hat{\lambda}}_y^O = \dot{\hat{\lambda}}_z^O = 0 \quad (6.79)$$

$$\begin{aligned}
0 &= \hat{\lambda}_x^O \sin \hat{\gamma}^O \cos \hat{\psi}^O + \hat{\lambda}_y^O \sin \hat{\gamma}^O \sin \hat{\psi}^O - \hat{\lambda}_z^O \cos \hat{\gamma}^O \\
&\quad - \hat{\lambda}_\psi^O \sin \hat{\phi}^O \sin \hat{\gamma}^O \frac{\hat{L}^O}{\cos^2 \hat{\gamma}^O}
\end{aligned} \tag{6.80}$$

$$0 = \hat{\lambda}_x^O \cos \hat{\gamma}^O \sin \hat{\psi}^O - \hat{\lambda}_y^O \cos \hat{\gamma}^O \cos \hat{\psi}^O \tag{6.81}$$

$$0 = \hat{\lambda}_\gamma^O \sin \hat{\phi}^O \hat{L}^O - \hat{\lambda}_\psi^O \cos \hat{\phi}^O \frac{\hat{L}^O}{\cos \hat{\gamma}^O}. \tag{6.82}$$

The TPBVP (6.73) – (6.82) has the same functional form as the TPBVP in (6.26) – (6.35). The initial conditions, for example, $\hat{x}^O(0) = x_{00}^O(0) + \epsilon_1 x_{10}^O(0)$ can be determined from (6.44) by integration. Since the right hand side of (6.44) is a known function, we obtain by integration

$$\begin{aligned}
x_{10}^O(0) &= \int_0^\infty [\cos(\gamma_{00}^{L_1} + \gamma_{00}^O) \cos(\psi_{00}^{L_1} + \psi_{00}^O) - \cos \gamma_{00}^O \cos \psi_{00}^O] d\tau_1 \\
&= \int_0^\infty [\cos(\gamma_{00}^{L_1} + \gamma_{00}^O) - \cos \gamma_{00}^O] d\tau_1 \\
&= \int_{-\gamma_{00}^O}^0 \frac{\cos(\gamma_{00}^{L_1} + \gamma_{00}^O) - \cos \gamma_{00}^O}{L_{00}^{L_1}} d\gamma_{00}^{L_1} \\
&= \frac{\sin \gamma_{sat} - \gamma_{sat} \cos \gamma_{00}^O}{k_1 \text{sign}(\gamma_{00}^O)} + \frac{4}{K} \left(\cos \gamma_{00}^O - \cos \frac{\gamma_{sat} + \gamma_{00}^O}{2} \right)
\end{aligned} \tag{6.83}$$

where γ_{sat} is defined by

$$\begin{aligned}
&\text{if } |L_{00}^{L_1}(0)| \leq 1 \quad \gamma_{sat} = 0 \\
&\text{if } L_{00}^{L_1}(0) > k_1 \quad K \sin \frac{\gamma_{sat} - \gamma_{00}^O}{2} = k_1 \\
&\text{if } L_{00}^{L_1}(0) < -k_1 \quad K \sin \frac{\gamma_{sat} - \gamma_{00}^O}{2} = -k_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_{10}^O(0) &= 0 \\
z_{10}^O(0) &= \frac{-\cos \gamma_{sat} + 1 - \gamma_{sat} \sin \gamma_{00}^O}{k_1 \text{sign}(\gamma_{00}^O)} + \frac{4}{K} \left(\sin \gamma_{00}^O - \sin \frac{\gamma_{sat} + \gamma_{00}^O}{2} \right).
\end{aligned} \tag{6.84}$$

Since $\hat{x}^O(0)$, $\hat{y}^O(0)$ and $\hat{z}(0)$ are known and the TPBVP (6.73) – (6.82) have the same functional form as those of the TPBVP (6.26) – (6.35), the slow costate variables can be computed by the same method described in the derivation of the zeroth order outer solution.

We remark that the right boundary layer terms are identically zero in this particular problem.

6.4 Implementation

Derivation of the Body Axis Roll Rate Command.

Since the equations of motion (6.1) – (6.6) are defined in the wind axis, all the initial conditions and the target velocity must be evaluated in the wind axis. Furthermore, since p_w in (6.6) is defined as the wind axis roll rate, it must be converted to the body axis roll rate in order to be used as the roll rate command to the autopilot. The wind axis roll rate can be expressed in terms of the body axis roll rate p_b as follows

$$p_w = \cos \beta \cos \alpha p_b + \sin \beta q_b + \cos \beta \sin \alpha r_b - \dot{\alpha} \sin \beta \quad (6.85)$$

where p_b , q_b and r_b denote the body axis roll rate, pitch rate and yaw. α and β are the angle of attack and the side slip, respectively.

Since β and r_b are negligibly small in a bank to turn missile, p_b can be expressed as follows

$$p_b \approx \frac{p_w}{\cos \alpha}. \quad (6.86)$$

Determination of the Control Weighting k_1 and k_2 .

The validity of the singular perturbation method depends on the insignificance of the higher order terms. Since $\epsilon_1 = \frac{V^2 k_1}{RL_{wmax}}$, as the final time is approached, the

missile approaches the target, R approaches zero and ϵ_1 increases. When ϵ_1 increases, the neglected higher order terms become significant, causing the approximation to no longer be valid. This is the reason for the failure of the usual singular perturbation method in the terminal phase. To avoid this problem, k_1 can be adjusted to cancel the effect of R going to zero. A simple solution is to pick k_1 so that ϵ_1 is constant. A second cause of failure for the singular perturbation method is the size of the coefficients multiplying ϵ_1 and ϵ_2 in the expansion. If these coefficients are very small, the solution will remain valid even as ϵ_1 and ϵ_2 get large. For example, if $\gamma(0) \approx \gamma_{00}^O$, then the higher order coefficients are all small so that larger values for ϵ_1 and ϵ_2 do not destroy the approximation. Due to modeling errors and unmodelled target motion, $\gamma(0)$ is not well approximated by γ_{00}^O and the fixed weight singular perturbation method fails.

CHAPTER VII SIMULATION RESULTS

7.1 Deterministic Simulation Results

The performance of the guidance law developed in the previous chapter will be demonstrated by using 6 DOF bank-to-turn missile simulation program developed at the University of Florida. Table 7.1 shows the missile characteristics used in the simulation program.

Table 7.1 Summary of Missile Characteristics

Initial velocity (Mach)	0.9
Boost thrust (lb)	4712
Burn out time (sec)	2.6
Initial weight (lb)	165
Final weight (lb)	115
Maximum vertical lift (g)	100
Maximum roll rate (deg/sec)	500

The simulated target is a three dimensional maneuvering target with the maximum acceleration $9g$. Once the range from the missile to the target is less than 6000 ft, the target initiates an instantaneous $9g$ evasive maneuver in a plane determined by the target roll angle which is determined by an input parameter. The Rockwell autopilot which uses a dithered signal in the yaw channel was used in the simulation program. There is also a 0.4 second delay between missile launch and autopilot action.

The miss distances of the singularly perturbed guidance law with the control weight adjusting scheme (WASP), with a constant control weight (WUSP) and the optimal linear guidance law (OL) are shown in Figure 7.1.

From Figure 7.1, it can be seen that the weight adjusted singularly perturbed guidance law developed in Chapter VI performs well under the various engagement scenarios in spite of the modelling errors such as the target maneuvering and neglected autopilot missile air frame dynamics.

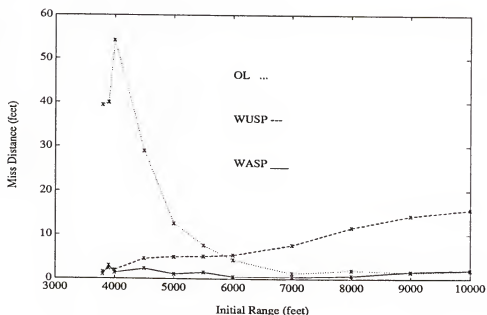


Figure 7.1: Miss distances

In order to illustrate the performance of the missile with the WASP control for an initial range of 5000 feet and initial gimbal elevation and azimuth angles of 0° and 40° , is shown the commanded vertical lift, roll rate and the corresponding trajectory are shown in Figure 7.2 – Figure 7.5.

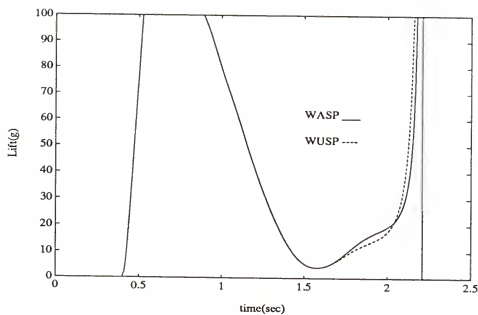


Figure 7.2: Vertical Lift Command

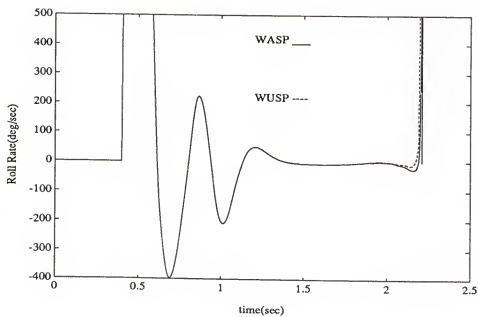


Figure 7.3: Roll Rate Command

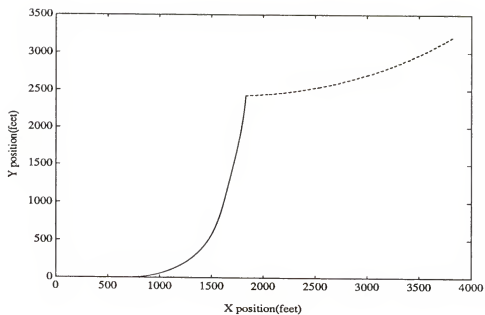


Figure 7.4: X-Y Plane Trajectory

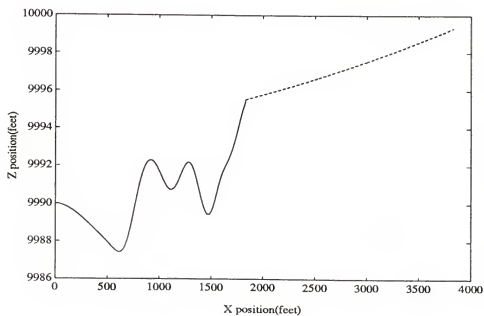


Figure 7.5: X-Z Plane Trajectory

From Figure 7.2, the vertical lift command is saturated during the end game. This comes from the fact that the LOS angle to the intercept point must be zero at the final time to satisfy the final conditions (the relative distances at the final time are zero). Hence, when the time-to-go is extremely small, the commanded vertical lift will be saturated even if the LOS angle is small.

In order to further compare the singularly perturbed guidance law with performance index adjustment with the optimal linear guidance law, we will examine some of the simulation results in more detail. The test trajectory chosen for illustration has an initial range of 10,000 feet and initial gimbal angles of 0° and 40° for elevation and azimuth. The behavior of the missile wind axis line of sight angle (LOS) to the predicted intercept point is shown in Figure 7.6 for the weight adjusted performance index (WASP) and the constant weight performance index (WUSP).

Both schemes keep the LOS fairly small, indicating good tracking, after the initial transient. However, the constant weighting control law fails to track in the end game due to the inability to track the unmodelled target maneuver. In all terminal guidance schemes, the time-to-go is a significant parameter which must be estimated. The estimate given by the range divided by the closing velocity has been shown by Riggs and others [39][40] to be in error for maneuvering targets and in the final portion of the trajectory. Riggs [39] suggested an improvement, known as Riggs algorithm, which takes the known missile acceleration into account. The estimated time-to-go in the weight adjusted singular perturbation is superior to Rigg's approximation as shown in Figure 7.7 where the actual time-to-go is compared with the two estimates.

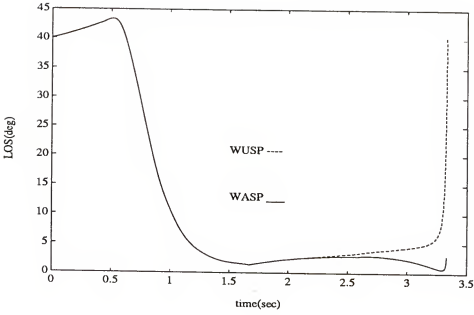


Figure 7.6: LOS Angle

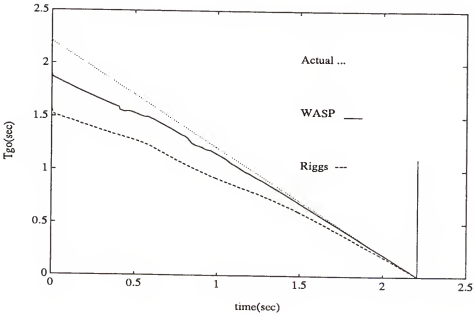


Figure 7.7: Time-To-Go

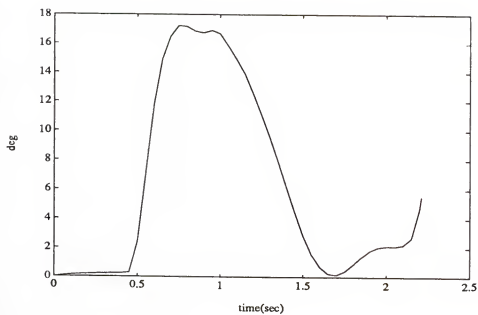


Figure 7.8: Angle of Attack

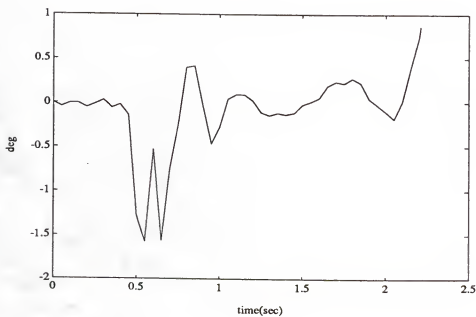


Figure 7.9: Side Slip

In Figure 7.7, all three curves were generated from the same trajectory, only the method for determining the time-to-go was modified. The improvements in the time-to-go estimates may be another reason for the superior performance of our new guidance law. The history of the angle of attack and the side slip for an initial range of 5000 feet are shown in Figure 7.8 and Figure 7.9. From Figure 7.9, WASP keeps the side slip within the limit ($\pm 5^\circ$) for a BTT missile.

7.2 Stochastic Simulation Results

So far, we assumed that the target velocity is known. Actually, the only available measurements with respect to the target are i) the range measurement, ii) the range rate measurement, iii) the elevation angle measurement and iv) the azimuth angle measurement when an active seeker is used. In this simulation, the extended Kalman filter was used to estimate the target velocity from the above four available measurements. The covariances of the seeker measurement noise used in the simulation are given in Table 7.2.

Table 7.2 Covariance of Measurement Noise

Type of Measurements	Covariance
Azimuth and Elevation Angle	$((1.5/R)^2 + 2.225^2 \times 10^{-10})/\Delta T$
Range	$(9 + 10^{-16}R^4)/\Delta T$
Range Rate	$(9 + 4 \times 10^{-22}R^4)/\Delta T$

R : the relative distance (ft) ΔT : the measurement step (sec)

The performance was determined by 10 runs for each initial condition. The mean and the variance of the final miss distances with the initial seeker elevation angle 0.107° and azimuth angle 40° are shown in Table 7.3

Table 7.3 Statistics of Miss-distances

Initial (ft)		Guidance law		
		WASP	WUSP	OL
10000	Mean	36.9	65.8	32.3
	Var	749.2	604.8	572.9
8000	Mean	24.0	44.8	24.5
	Var	130.1	302.1	101.5
7000	Mean	22.3	32.8	24.6
	Var	126.5	107.1	159.2
5000	Mean	30.8	30.9	34.8
	Var	435.7	440.0	591.3
4500	Mean	31.5	29.7	58.1
	Var	309.4	186.4	285.8
4000	Mean	41.9	44.3	75.3
	Var	268.5	328.6	813.4
3800	Mean	37.8	38.0	61.3
	Var	295.8	339.4	701.8

A typical trajectory and command history of an initial range 5000 feet are shown in Figure 7.10 – Figure 7.13.

The average performance of the new weight adjusted singular perturbation control law is superior to the constant weight singular perturbation and optimal linear guidance laws. The relatively large sample variance observed for the miss-distances indicates more runs are required for a more thorough evaluation.

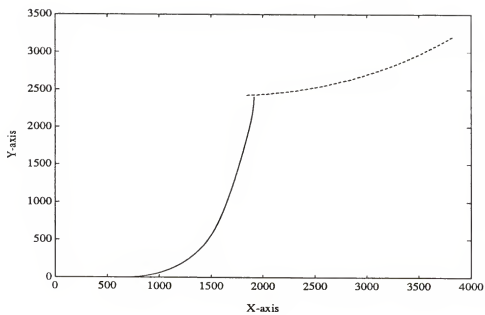


Figure 7.10: X-Y Plane Trajectory

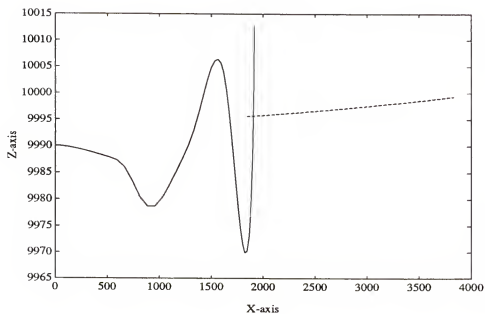


Figure 7.11: X-Z Plane Trajectory

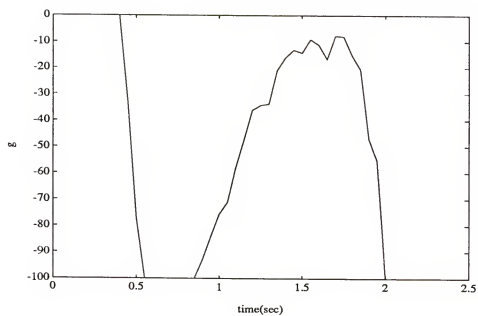


Figure 7.12: Vertical Lift Command

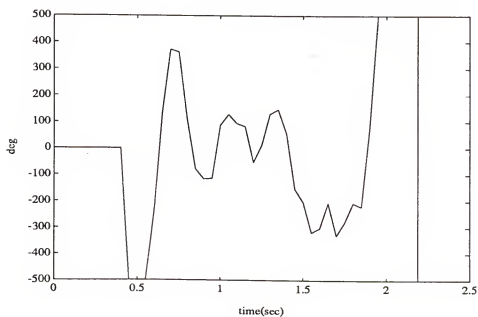


Figure 7.13: Roll Rate Command

CHAPTER VIII

CONCLUSIONS

In this dissertation, the singular perturbation method was considered in order to analyze a nonlinear TPBVP. Particular emphasis has been given to a nonlinear TPBVP arising in a nonlinear optimal control problem with free final time. The dissertation consists of two main parts.

In the first part, an explicit method to construct a uniformly valid asymptotic solution of the full problem was developed. By our construction method based on a recursive scheme, the solution of an optimal control problem can be approximated to any order. With the assumptions imposed on the reduced problem, if ϵ and the zeroth order boundary layer jump δ_0 are sufficiently small, the solution of the full problem exists and approaches the asymptotic solution as ϵ goes to zero. In the proof of this result, one of the crucial assumptions is an eigenvalue criterion assumed in Assumption A.5. However, as shown in Sacker's work [41], eigenvalues with zero real parts may be allowed if a form of conditional stability is satisfied so that Assumption A.5 could be weakened.

In constructing an asymptotic solution, one of the most difficult things is to obtain the zeroth order boundary layer solution. However, in an actual application problem, sometimes it may be sufficient to obtain the zeroth order asymptotic solution. In this case, we can use an approximate zeroth order boundary layer solution obtained from the linearized boundary layer problem with respect to the zeroth order outer solution.

The singular perturbation method can also be applied to a system with multiple time scale property on an infinite time interval [42]. One of the interesting questions in this area is that if a reduced order system is asymptotically stable, under which conditions a full system is asymptotically stable? Recently, this question was partly answered [43][44]. In the robustness analysis in a control problem with neglected higher order dynamics, the singular perturbation will be a powerful tool.

In the second part, the singular perturbation method was applied to an air-to-air BTT guidance problem. In order to compensate for model errors and target motion, a control weight adjusting scheme is used to derive a guidance law. In both the deterministic and the stochastic cases, this scheme gives a considerable improvement over an optimal linear guidance law when the initial range is short. When the initial range is long (above 7000 ft), this scheme also works well while the usual singular perturbed guidance law with constant control weight does not work well. The computation is simple enough so that it can be implemented by on board computation.

The higher gain control resulting from the smaller control weighting forces the missile to quickly reach a coasting arc where the vertical lift command is zero. If we assume that i) the current missile position is on the coasting arc, ii) the missile roll angle can be changed instantaneously and iii) the future behavior of the target is unknown, the center of the target reachable set will be placed on the center of the missile reachable set. Thus, the idea of the high gain control implicitly coincides with the idea of the reachable set control [45].

The performance of the guidance law is heavily dependent on the target velocity. Since the covariance of the measurement noise in the elevation and azimuth angle is

larger as the missile approaches closer to the target due to glint noise, the estimated value of the target velocity is not good during the terminal phase. This leads to an increased miss distance. In order to further reduce the miss distance, an improved estimation scheme for bearing only measurements should be developed.

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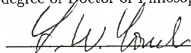
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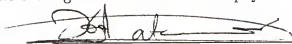
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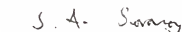
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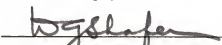
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